You have 2 hours. Total possible score is 100 points. The exam consists of 6 problems on 4 pages
Please show all work. Be sure to define your terms and explain your reasoning and logic, particularly if you don’t complete all of the algebra or calculus in a problem. The examination is closed book except for the two formula sheets provided previously.

Useful information: \( \cos(a-b) - \cos(a+b) = 2 \sin a \sin b \)
First three Legendre polynomials are: \( P_0(x) = 1; \ P_1(x) = x; \ P_2(x) = \frac{1}{2} (3 x^2 -1) \)

First few spherical harmonics:

\[
Y_{00} = \frac{1}{\sqrt{4\pi}} \\
Y_{11} = -\frac{3}{\sqrt{8\pi}} \sin \theta \ e^{i\phi}, \quad Y_{10} = \frac{3}{\sqrt{4\pi}} \cos \theta \\
Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta \ e^{2i\phi}, \quad Y_{21} = -\frac{15}{\sqrt{8\pi}} \sin \theta \cos \theta \ e^{i\phi}, \quad Y_{20} = \frac{5}{\sqrt{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \\
Y_{l,m}(\theta, \phi) = (-1)^m Y_{l,-m}^* (\theta, \phi) \\
\int_0^\infty r^n e^{-r} dr = n! , \text{ for } n \text{ integral}
\]

1. (10 points)
(a) Write down the complete Maxwell’s equations in media in terms of the vector fields \( \vec{E}, \vec{D}, \vec{B}, \text{and } \vec{H} \).
(b) Write down the boundary conditions satisfied by the fields \( \vec{E}, \vec{D}, \vec{B}, \text{and } \vec{H} \) at an interface. Explain the procedure for deriving these boundary conditions starting from Maxwell’s equations. Note, however, that you do NOT have to actually derive the boundary conditions.
(c) Now consider the case of the fields in vacuum. Rewrite Maxwell’s equations in terms of the vector fields \( \vec{E} \) and \( \vec{B} \) in vacuum.

2. (10 points) An infinitely long wire is bent into the hairpin (2 straight line segments plus a half-circle) shape shown in the figure. Find an exact expression for the magnetic field at the point \( P \) which lies at the center of the half-circle of radius \( a \).
3. (15 points)  \( \rho(x) = r^3 e^{-r} \sin^2 \theta \cos 2\phi. \)

Make a multipole expansion of the potential due to this charge density and determine all of the nonvanishing multipole moments in spherical coordinates. Write down the potential at large distances as a finite expansion in spherical harmonics.

4. (20 points) Consider a spherical shell of inner radius \( a \) and outer radius \( b \), made of material of magnetic permeability \( \mu \). Place this shell into a region of formerly uniform constant magnetic induction \( B_0 \).

(a) Use Maxwell’s equations for this problem to show that a scalar potential can be used to do this problem. What differential equation does this scalar potential satisfy?

(b) Write series expansions for the potential in the three regions \( r<a \), \( a<r<b \), and \( r>b \).

(c) Write the boundary conditions on the fields at the interfaces. Explain how you would take derivatives of the magnetic scalar potential to write a set of simultaneous equations for the coefficients in the series. Explain which coefficients you expect to be nonzero and why. YOU DO NOT NEED TO FIND EXPLICITLY THE SET OF SIMULTANEOUS EQUATIONS OR TO SOLVE FOR THE COEFFICIENTS.

(d) This problem is the example of magnetic shielding from the textbook. Without calculation, draw a sketch of the lines of \( B \) in the region near a spherical shell with very high magnetic permeability.
5. (25 points) Use the following Green’s function to find the solutions to the problem below. One form of the Dirichlet Green’s function for the inside of a grounded cylindrical box defined by the surfaces $z=0$, $z=L$, $\rho=a$ is:

$$G(\tilde{x}, \tilde{x}') = \frac{4}{L} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{i n \pi \rho / L} \frac{\sin(n \pi z / L)}{\sin(n \pi z' / L)} I_m \left( \frac{n \pi \rho}{L} \right) K_m \left( \frac{n \pi \rho}{L} \right) - K_m \left( \frac{n \pi \rho}{L} \right) I_m \left( \frac{n \pi \rho}{L} \right)$$

YOU MAY LEAVE INTEGRALS (over Bessel Functions) IN YOUR ANSWER TO THIS PROBLEM.

(a) The walls of the conducting cylindrical box are all at zero potential, except for a disc in the upper end, defined by $\rho=b$, $b<a$, which is at potential $V$. Find an expansion for the potential inside the cylinder.

(b) Suppose now that a total charge $Q$ is uniformly distributed on an infinitesimally thin cylindrical shell located at $\rho=c$, where $c<b<a$, which extends from $z=0$ to $z=L$ inside the cylinder. First write an expression for the charge density in cylindrical coordinates using delta functions. Then derive an expression for the additional term in the potential in the presence of this line charge.
6. (20 points) Two conducting planes at zero potential meet along the z axis, making an angle \( \beta \) between them, as shown in the figure. The different parts of this problem will allow you to explain the steps in the derivation of the 2D Dirichlet Green’s function \( G(\rho, \phi; \rho', \phi') \) in the region \( 0 \leq \rho \leq \infty, \ 0 \leq \phi \leq \beta \), which is given by the infinite series:

\[
G(\rho, \phi; \rho', \phi') = 4 \sum_{m=1}^{\infty} \frac{1}{m} \frac{\rho}{\rho'} \sin\left(\frac{m\pi\rho}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right).
\]

Recall that the Laplace equation in 2D is:

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0,
\]

and it has solutions of the form \( \Phi = \left[ \rho^\nu \right] \left[ \cos \nu \phi \right] \). Also recall that in 2D, the delta function \( \delta(\ddot{x} - \ddot{x}') = \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi') \). 

(a) Find a series expansion for \( \delta(\phi - \phi') \) in terms of functions which satisfy the boundary conditions for this Green’s function.

(b) Use this series to write an ansatz (i.e., educated guess) for the Green’s function \( G \) as a product of a function of \( \rho, \rho' \) times a function of \( \phi, \phi' \).

(c) Explain how you could use your guess for \( G \) to obtain the differential equation which is satisfied by the function \( R \). (You do NOT need to derive the actual differential equation, however.)

(d) Given the information in the box above, write your guess for a solution for \( R(\rho, \rho') \) a constant \( C \) times a function which satisfies the boundary conditions of this problem and also symmetric in \( \rho \) and \( \rho' \).

(e) Explain in words the general procedure for finding the constant \( C \). (You do NOT need to write out the actual equations.)

Note that application of this condition would give the answer \( C = \frac{2\beta}{m} \). You should NOT derive this, however. Putting all of these parts together, you could obtain the given solution for the Green’s function.