Analytic Functions of Complex Variables

\[ z = x + iy \]
\[ f(z) = u + iv \]

An analytic function has \( \frac{df}{dz} \) well defined.

\[ \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \]

with the limit independent of the particular approach to \( z \).

\[ \frac{df}{dz} = \frac{du + iv}{\Delta x + i\Delta y} \]

\[ \frac{\Delta x}{\Delta z} = \frac{\Delta y}{\Delta z} \]

\[ \lim_{\Delta z \to 0} \frac{\Delta x}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta y}{\Delta z} = \frac{\Delta u}{\Delta x} = \frac{\Delta v}{\Delta y} \]

Set real and imaginary parts equal to one another in these 2 equations.

\[ \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{Cauchy-Riemann condition} \]

necessary for existence of \( \frac{df}{dz} \)

\[ \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \nabla^2 u = 0 \quad \text{in 2D, i.e. } u \text{ is a harmonic function of Laplace's equation} \]

Also,

\[ \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = \nabla u \cdot \nabla v = 0 \]

If \( u(x, y) = C \), \( du = 0 \) on curve \( \Rightarrow \nabla v = \partial u / \partial x \), \( dx \) on curve \( u(x, y) = C \). \( \perp \) curve

So curves with \( u = C_1 \) are orthogonal to curves with \( v = C_2 \).

If \( u = C_1 \) is line of \( E \) field, then \( v = C_2 \) is equipotential surface.

Actually, this is true because \( \vec{E} = -\nabla \phi \)

Example 1
\[ f(z) = e^z = e^{x+iy} = e^x \cos y + ie^x \sin y \]

Then \( u \) and \( v \) each satisfy \( \nabla^2 u = \nabla^2 v = 0 \).

Example 2
\[ f(z) = \frac{1}{z^2} = \frac{e^{-2i\phi}}{p^2} \Rightarrow u = \cos 2\phi \]

Example 3
\[ \vec{u} = (u_x, u_y) \]
\[ \vec{v} = (v_x, v_y) \]

Then Cauchy-Riemann condition \( \Rightarrow \vec{u} = (0, 1) \vec{v} \)

Corresponds to multiplying by \( -i \)

since \( -i(x+iy) = -ix + iy \)

Thus \( \vec{u} \perp \vec{v} \Rightarrow u \perp v \).