Physics 9HE-Modern Physics
Final Examination
March 18, 2015
(100 points total)
You may tear off this sheet.

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Miscellaneous data and equations:
\( c = 3.00 \times 10^8 \text{ m/s} \quad e = 1.60 \times 10^{-19} \text{ C} \quad 1 \text{ eV} = 1.60 \times 10^{-19} \text{ J} \quad 1 \text{ Å} = 10^{-10} \text{ m} \)
\( M_{\text{Sun}} = 2 \times 10^{30} \text{ kg} \quad M_{\text{Earth}} = 5.98 \times 10^{24} \text{ kg} \quad r_{\text{Earth}} = 6.38 \times 10^6 \text{ m} \)
\( m_e = 9.1094 \times 10^{-31} \text{ kg} \quad m_e = 0.5110 \text{ MeV/c}^2 \quad m_p = 1.6726 \times 10^{-27} \text{ kg} \quad 938.27 \text{ MeV/c}^2 \)
\( m_n = 1.6749 \times 10^{-27} \text{ kg} \quad 939.57 \text{ MeV/c}^2 \quad m = (1H) = 1.0078 \text{ u} \quad G = 6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2 \quad g = 9.81 \text{ m/s}^2 \quad \sigma = 5.67 \times 10^{-8} \text{ W}\cdot\text{m}^{-2}\cdot\text{K}^{-4} \)
\( h = 6.63 \times 10^{-34} \text{ J}\cdot\text{s} \quad h = \hbar/2\pi \quad k_B = 1.38 \times 10^{-23} \text{ J}\cdot\text{K}^{-1} \quad a_0 = 0.529 \text{ Å} \)
\( N(t) = N_0 \exp(-t/\tau_0) = N_0 \exp(-0.693t/\tau_1/2) \quad \kappa_c = 1/(4\pi\sigma_0) = 8.98 \times 10^9 \text{ m}^2/\text{C}^2 \quad R_H = 1.09678 \times 10^7 \text{ m}^1 \)
\( \gamma = 1 - \frac{v^2}{c^2} = 1 + \frac{0.5b^2}{c^2} \quad \text{if } v \ll c \quad \beta = \frac{v}{c} = [(\gamma^2 - 1)/\gamma]^{1/2} \)
\( T = \gamma T_0 \quad L = L_0/\gamma \quad \nu = \nu_0 \left(\frac{1 + \beta}{1 + \beta^2}\right)^{1/2} \quad \nu_0 = [\nu(1 - \beta)]^{1/2} \quad \text{for } \beta < 1 \quad \nu = c/\lambda \quad \nu \text{ (lect.) } = f(\text{book}) \)
\( \bar{p} = \gamma m \bar{u} \quad E = \gamma mc^2 = K + mc^2 \quad E^2 = p^2 c^2 + m^2 c^4 \)
\( E = pc = \hbar \nu \quad \Delta \nu = \Delta T \quad \frac{GM}{c^2} \left[ \frac{1}{r^2} - \frac{1}{r^2} \right] \approx -\frac{gH}{c^2} \quad \gamma = \Delta \nu \quad \beta = \left(\frac{\Delta \nu}{\nu} \right) \quad \frac{R_{\text{Sch}}}{c^2} = \frac{2GM}{c^2} \)
\( \lambda_{\max}T = 2.898 \times 10^{-3} \text{ m-K} \quad \bar{L} = \frac{2\pi c h}{\lambda^5} \quad \frac{1}{e^{\frac{hc}{kT}} - 1} \quad R(T) = \varepsilon\sigma T^4 \)
\( h\nu = K_{\text{max}} + \phi \quad n\lambda = 2d\sin\theta \quad F_{\text{coul}} = \frac{k_q q_z}{r^2} \quad F_{\text{radial}} = \frac{mv^2}{r} \)
\( \lambda = h/\pi \quad p = mk \quad \Delta x \Delta p_x \geq \hbar/2 \quad \Delta E \Delta t \geq \hbar/2 \quad v_{\phi} = \bar{a}/\bar{k} \quad v_{\varphi} = d\omega/dk \)
\( e^{ix} = \cos x + i \sin x \quad \cos x = \frac{1}{2} \left[ e^{ix} + e^{-ix} \right] \quad \sin x = \frac{1}{2i} \left[ e^{ix} - e^{-ix} \right] \)
\( \sin 2t = 2 \sin t \cos t \quad \cos 2t = \cos^2 t - \sin^2 t = 2 \cos^2 t - 1 = 1 - 2 \sin^2 t \quad \psi(x) = \frac{a_0}{2} + \sum_n a_n \cos(\frac{2\pi n}{\ell} x) + \sum_n b_n \sin(\frac{2\pi n}{\ell} x), \text{ with } n = \frac{2\pi n}{\ell} \text{, and} \quad a_0 = \text{aver. of } \psi \text{ over } \ell, \quad a_n = \frac{1}{\ell} \int_0^\ell \psi(x') \cos(\frac{2\pi n}{\ell} x')dx', \quad b_n = \frac{1}{\ell} \int_0^\ell \psi(x') \sin(\frac{2\pi n}{\ell} x')dx' \)
\( \psi(x) = \int \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k)e^{ikx}dk, \text{ with } c(k) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(x')e^{-ikx'}dx' \)
\( \hat{H}\psi = i\hbar \frac{\partial \psi}{\partial t} \quad \hat{H} = \hat{K} + V \quad \psi = \psi e^{-i\hat{\psi}/\hbar} = \psi e^{-i\hat{\psi}} \quad \hat{H}\psi = E\psi \)
\( \hat{p}_x = -i\hbar \frac{\partial \psi}{\partial x} \quad \hat{K}_x = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \quad \Delta \lambda = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \quad \langle A \rangle = \int \psi \hat{A}\psi dx = \int \psi \hat{A}\psi^* dx \quad \hat{A}\psi = a\psi \)
\[ k = \sqrt{\frac{2m(E-V)}{\hbar^2}} \quad \psi \propto e^{ikx} \quad \psi \propto \sin kx, \cos kx \quad \kappa = \alpha = \frac{1}{\delta} = \sqrt{\frac{2m(V-E)}{\hbar^2}} \quad \psi \propto e^{ikx} \]

\[ \psi_n = \left(\frac{2}{L}\right)^{1/2} \sin \left(\frac{n\pi x}{L}\right) \quad E_n = \frac{n^2\hbar^2 n^2}{2mL^2} \]

\[ \psi_n = H_n(x)e^{-ax^2/2} \quad \alpha = \sqrt{\frac{mk}{\hbar^2}} \quad \alpha = \sqrt{\frac{\kappa}{m}} \quad E_n = \left(n + \frac{1}{2}\right)\hbar\omega \]

\[ \psi = \sum_{n\lambda} c_{n\lambda} \phi_{n\lambda}(\vec{r}) \quad \Psi_\alpha(x) = u_\alpha(x)e^{ikx}, \text{where} \quad u_\alpha(x) = u(x + A) \]

\[ \begin{align*}
  V &= \sum_{i=1}^{n} \left(\frac{+ \text{ or } -n_i q_i e^2}{4\pi \varepsilon_0 \rho_i}\right) + Ae^{-r_i^2/\rho_i} \\
  F_{MB} &= A \exp(-E/k_B T) \\
  F_{BE}(E) &= \frac{1}{B_2 \exp(E/k_B T) - 1} \\
  F_{FD}(E) &= \frac{1}{\exp\left([E-E_F]/k_B T\right) + 1} \\
  \lambda_{max} &= 2.898 \times 10^{-3} m-K \\
  l(\lambda, T) &= \frac{2\pi c^3 \hbar}{\lambda^2} \cdot \frac{1}{e^{\lambda c T} - 1} \\
  R(T) &= \frac{c_0}{A^{1/4}} \\
  E_F &= \frac{\hbar^2 k_F^2}{2m_e} \\
  g(E) &= \frac{(2m_e)^{3/2}}{2\pi^2 \hbar^2} E^{1/2} \\
  k_F &= (3\pi^2 n_{\text{coll}})^{1/3} \\
  v_F &= \frac{\hbar(3\pi^2 n_{\text{coll}})^{1/3}}{m_e} \\
  B(\vec{r}) &= [N_m + Z_m(\vec{r}) - M(\vec{r})]e^2 \quad a = a_A A^{1/3} - 3 \frac{Z(Z-1)e^2}{5} \frac{1}{4\pi^2 R_{\text{nucl}}} - a_s \frac{(N-Z)^2}{A} + \delta \\
  R_{\text{nucl}} &= (1.2 \times 10^{-15} m) A^{1/3} \\
  Q &= [M_{\text{initial}} - M_{\text{final}}]c^2
\end{align*} \]

--- Tear off this sheet and begin exam ---
Physics 9HE-Modern Physics  
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(100 points total)

Name (printed)_______________________________________

Name (signature)_____________________________________

Student ID No.____________________________________

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[1] (70 points) Answer the following ten independent questions briefly:

(a) (5 points) State the fundamental postulate of General Relativity.

Principle of Equivalence: No experiment can be done in a confined space that can detect the difference between a uniform gravitational field and an uniform acceleration. Or any nearly equivalent statement. (Partial credit for answering with one of the observable verification of this.)

(b) (9 points) The Space Station travels at an altitude of about 370 km and at a speed of 28,000 km/hour and has a clock inside. Assume that the gravitational force is essentially constant from the ground to this altitude and calculate the fractional changes in this clock’s time as observed on the ground, including both Special and General Relativity, and specify the sign of each change (faster, slower than a ground based clock).
(c) (7 points) Sketch the energy level system for a semiconductor which has been p doped, labelling the various electronic states and bands, the position of the Fermi level, and indicating the likely bonding and anti-bonding character of the bands involved as appropriate.

(d) (7 points) (Name the four fundamental forces and the particles which mediate them, and indicate what they are responsible for in nature (i.e. give an example of what they cause).

Any summary like that below, with full details in the table from text and lecture.
(e) (9 points) (Two parts) Consider the rotational excitations of a molecule of HCl, with atomic masses of $M_H = 1.0078 \text{ u}$ and $M_{Cl} = 34.9688 \text{ u}$ and a bond length of 1.26 Å.

(i) (4 points) What frequency of light would be necessary to excite this molecule from its ground state wave function $\Psi_{00}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{1}{\pi}}$ to a first excited state wave function $\Psi_{11}(\theta, \phi) = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta \exp(i\phi)$?

(ii) (5 points) Consider now light with polarization along the $x = rsin\theta cos\phi$ direction and write down the integrals that would have to be evaluated in order to determine whether this transition from the ground state to the first excited states would be allowed.
(f) (5 points) Write down all the eigenfunction relationships satisfied by the hydrogenic wave function $\Psi_{321}(r, \theta, \phi)$ and give the value of each eigenvalues in terms of fundamental constants. (Full credit for energy plus any two of the others.)

\[
\hat{\mathbf{E}}_{321} = E_3 \Psi_{321}, \text{ where } E_3 = -\frac{(13.6 \text{ eV})}{(3)} = -1.51 \text{ eV}
\]

\[
\hat{\mathbf{L}}_{321} = L (2 + 1) \Psi_{321}, \text{ where } L = t(2) + (3) = 6 t^2 
\]

\[
\hat{\mathbf{L}}_{2} \Psi_{321} = t(1) \Psi_{321}, \text{ where } L_2 = t(1) = t 
\]

\[
\hat{\mathbf{1}} \Psi_{321} = (-1)^2 \Psi_{321} = (+1) \Psi_{321}, \text{ even parity }
\]

(g) (7 points) Would the following be a suitable approximate wavefunction for two neutrons which overlap in space, where $a$ and $b$ represent two different sets of quantum numbers including the neutron spin and $C$ is a suitable normalization constant?

\[
\Psi(\vec{r}_1, \vec{s}_1; \vec{r}_2, \vec{s}_2) = C [\Phi_a(\vec{r}_1, \vec{s}_1) \Phi_b(\vec{r}_2, \vec{s}_2) + \Phi_b(\vec{r}_1, \vec{s}_1) \Phi_a(\vec{r}_2, \vec{s}_2)]
\]

Explain why or why not.

(\text{Because for anti-symmetry required of } \Psi, \text{ since neutrons are fermions.})

\[
= \Psi(\vec{r}_1, \vec{s}_1; \vec{r}_2, \vec{s}_2) \Rightarrow \text{Symmetric, so not ok for neutrons}
\]

(h) (7) The $\Sigma^+$ particle is composed of two up quarks "u" (each with mass $360 \text{ MeV/c}^2$ and charge $2e/3$) and a strange quark "s" (with mass $540 \text{ MeV/c}^2$ and charge $-e/3$). Estimate the $\Sigma^+$ mass (assuming weak binding) and calculate its charge. The actual $\Sigma^+$ mass is $1189.4 \text{ MeV/c}^2$. Does the assumption of weak binding seem correct?
So actual mass is only about 6% lower than sum of constituents, so weak binding assumption looks correct.
(i) (7 points) For a four-level laser system such as He-Ne, what conditions on level populations and state lifetimes is required for successful operation? Explain briefly with a level diagram.

Don't need all these words for full credit, just general idea.

(j) (7 points) The halflife of $^{14}_6C$ is 5730 years. If the fraction of this nuclide in the atmosphere is assumed to be constant over time at 1.55 parts per trillion ($1.55 \times 10^{-12}$), what would be the fraction in wood from a tree that died 8595 years ago?

$8595/5730 = 1.50$, so the tree has been removed from $^{14}_6C$ production for 1.5 halflives, and from the given relationship

$N(t) = N_0 \exp(-t/\tau_0) = N_0 \exp(-0.693t/\tau_{1/2})$,

we have, noting if you are a subtle thinker that, since the fraction is so small, the decay of $^{14}_6C$ does not significantly change the other isotope numbers, so $N$ and $N_0$ above can be replaced by fractions as

$F(8595) = F(0) \exp(-0.693(1.50)) = (1.55 \times 10^{-12}) \exp(-0.693(1.50)) = 5.48 \times 10^{-13}$.

(i) (7 points) The iso-probability contour of one of the molecular wavefunctions of CO is shown below, with the sign of the wavefunction also indicated. Show with a sketch and an equation the approximate atomic orbital makeup of this wave function, and indicate whether it is bonding or anti-bonding.
More than you ever wanted to know here is from Slide 20 of Set 6 (next page), with the equation for what is called $1\pi_x$ here being

$$\varphi_{1\pi_x} \approx C_{2p_x} \varphi_{C2p_x} + C_{O2p_x} \varphi_{O2p_x}$$

and $C_{O2p_x}$ clearly greater than $C_{C2p_x}$

The LCAO or tight-binding picture for CO:

Chemist's picture (no core):

$$\varphi_{j}^{\text{MO}} \left( \mathbf{r} \right) = \sum_{\text{Atoms } A, \text{Orbitals } i} c_{A,j} \varphi_{A}^{\text{AO}} \left( \mathbf{r} \right)$$

Atomic orbital makeup

15. Carbon Monoxide

Symmetry: $C_{\text{inn}}$

- $x^2 - y^2$ NEG.
- $xy$ POS.
- Isocyanurates of $\phi_{2s}$

- Bonding $2\pi$
  - $\left( s, 2 \right)$

- Bonding $3\pi$
  - $\left( s, 3 \right)$

- Bonding $1\pi$
  - $\left( x, 2 \right)$
  - $\left( y, 2 \right)$

- Bonding $4\sigma$
  - $\left( x^2 - y^2, 4 \right)$

- Bonding $3\sigma$
  - $\left( x^2 - y^2, 3 \right)$

- Non/Weakly Bonding $4\sigma$
  - $\left( x^2 - y^2, 4 \right)$

- Non/Weakly Bonding $3\sigma$
  - $\left( x^2 - y^2, 3 \right)$

Core:

- $C_{\text{core}}: \left( \begin{array}{c} x^2 - y^2 \end{array} \right)$
- $O_{\text{core}}: \left( \begin{array}{c} x^2 - y^2 \end{array} \right)$

Chemist's picture (no core):

- $C_{\text{core}}: \left( \begin{array}{c} x^2 - y^2 \end{array} \right)$
- $O_{\text{core}}: \left( \begin{array}{c} x^2 - y^2 \end{array} \right)$
[2] (20 points) Separation of variables—As an approximation to the quantum corral shown below, consider a particle moving in two dimensions $r$ and $\phi$ in a cylindrical infinite potential well of radius $r_0$, as shown below:

$$V = \infty$$
$$For \ r \geq r_0$$

![Image of 48 iron atoms on a Cu(111) surface—a “quantum corral”](image)

The kinetic energy operator in this coordinate system can be shown to be:

$$\hat{K} = -\frac{\hbar^2}{2m} \nabla^2 = -\frac{\hbar^2}{2m} \left[ \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right]$$

(a) (5 points) What are the boundary conditions on the wave function and its derivative for this problem? Be careful, these are special.

Neither $\psi$ nor $d\psi/dr$ is continuous at $r = r_0$, since $V$ is infinite. Thus, $\psi(r > r_0)$ and $d\psi/dr (r > r_0) = 0$, but we really only need the first one to solve the problem, just like in the particle in the rigid box.

(b) (15 points) Set up the Schroedinger Equation for this problem inside the box, and show that it can be separated into two differential equations in $r$ and $\phi$, but you need not solve either one.
Let \( \Psi(r, \phi) = R(r) \Phi(\phi) \), and substitute

\[
\hat{K}[R(r)\Phi(\phi)] = \frac{-\hbar^2}{2m} \left[ \frac{\Phi(\phi)}{r} \frac{d^2 R(r)}{dr^2} + \Phi(\phi) \frac{d^2 R(r)}{r^2 d\phi^2} + \frac{R(r)}{r} \frac{d^2 \Phi(\phi)}{d\phi^2} \right] = ER(r)\Phi(\phi)
\]

Now divide by \( R(r)\Phi(\phi) \) to yield

\[
\frac{-\hbar^2}{2m} \left[ \frac{1}{r} \frac{d R(r)}{dr} + \frac{1}{R(r)} \frac{d^2 R(r)}{r^2 dr^2} + \frac{1}{\Phi(\phi)} \frac{d^2 \Phi(\phi)}{d\phi^2} \right] = E
\]

\[
-\frac{\hbar^2}{2m} \left[ \frac{r}{R(r)} \frac{d R(r)}{dr} + \frac{r^2}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{1}{\Phi(\phi)} \frac{d^2 \Phi(\phi)}{d\phi^2} \right] = r^2 E - \frac{\hbar^2}{2m} \left[ \frac{1}{\Phi(\phi)} \frac{d^2 \Phi(\phi)}{d\phi^2} \right] = 0
\]

The only way this can be true for all \( r \) and \( \phi \) is for red and blue to equal the same constant \( C_4 \) with opposite sign.

\[
\frac{1}{\Phi(\phi)} \frac{d^2 \Phi(\phi)}{d\phi^2} = \text{constant} = C_4
\]

\[
\alpha \frac{d^2 \Phi(\phi)}{d\phi^2} = C_4 \Phi(\phi) \quad \text{--- Diff. Eqn. 1, which yields } C_4 = \text{m}^2 \text{ and same } \Phi \text{ functions as H atom!!}
\]

\[
\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \text{ with } m = 0, \pm 1, \pm 2, ...
\]

which also implies that

\[
-\frac{\hbar^2}{2m} \left[ \frac{r}{R(r)} \frac{d R(r)}{dr} + \frac{r^2}{R(r)} \frac{d^2 R(r)}{dr^2} \right] - r^2 E = -\frac{\hbar^2}{2m} C_4
\]

\[
\alpha \frac{d^2 R(r)}{dr^2} + 2 \frac{d R(r)}{dr} + \frac{2m}{\hbar^2} E(r)R(r) - m^2 R(r) = 0 \quad \text{--- Diff. Eqn. 2, Bessel's equation}
\]

Full credit if you get to the equation with red and blue in it.

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[3] (10 points) Nuclear decay--

We have heard much in the last few years about the problems with the nuclear reactors in Japan, and in particular the problems associated with \( ^{131}_{53}I \), an isotope of iodine, which is selectively incorporated into the thyroid gland, possibly causing cancer.

(a) (2 points) What is X above?

\[ X = 131 - 53 = 78 \]

(b) (5 points) What is the binding energy per nucleon of \( ^{131}_{53}I \) if its total mass is 130.9061 u?
Binding energy per nucleon = \( \frac{1}{131} [53m_{^1H} + 78m_{^1N} - m_{^{131}I} ] \)

\[
\begin{align*}
= \frac{1}{131} [53 \times 0.0078 \quad 931.494 \quad \text{MeV} \quad 7\times 39.57 \quad \text{MeV} \quad 13\times 0.90 \quad 6\times 31.494 \quad \text{MeV} \quad ] c^2 \\
= \frac{1}{131} [49754.26 + 73286.46 - 121938.25] \text{MeV} = 1102.47 / 131 = 8.41 \text{ MeV/nucleon}
\end{align*}
\]

Which agrees in magnitude with the values given in Thornton and Rex, Fig. 12.6.

(c) (3 points) \(^{131}I_{53} \) decays via \( \beta^- \) with a halflife of 8.02 days, with the \( \beta^- \) exposure being cancer producing. Write down the decay scheme for this process, including all particles involved, assuming that the chemical symbol for the atom produced is Xe.

\[ ^{131}I_{53} \rightarrow ^{131}Xe_{54} + \beta^- + \bar{\nu}_e \]