Miscellaneous data and equations:

- \( c = 3.00 \times 10^8 \text{ m/s} \)
- \( e = 1.60 \times 10^{-19} \text{ C} \)
- \( 1 \text{ eV} = 1.60 \times 10^{-19} \text{ J} \)
- \( 1 \text{ Å} = 10^{-10} \text{ m} \)
- \( M_{\text{Sun}} = 2 \times 10^{30} \text{ kg} \)
- \( M_{\text{Earth}} = 5.98 \times 10^{24} \text{ kg} \)
- \( r_{\text{Earth}} = 6.38 \times 10^6 \text{ m} \)
- \( m_e = 9.1094 \times 10^{-31} \text{ kg} \)
- \( m_p = 1.6726 \times 10^{-27} \text{ kg} \)
- \( m_n = 1.6749 \times 10^{-27} \text{ kg} \)
- \( m(1H) = 1.0078 \text{ u} \)
- \( G = 6.67 \times 10^{-11} \text{ N}\text{m}^2/\text{kg}^2 \)
- \( g = 9.81 \text{ m/s}^2 \)
- \( \sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4} \)
- \( h = 6.63 \times 10^{-34} \text{ J}\text{s} \)
- \( k_B = 1.38 \times 10^{-23} \text{ J/K}^{-1} \)
- \( a_0 = 0.529 \text{ Å} \)

\[ N(t) = N_0 \exp(-t/\tau_0) = N_0 \exp(-0.693t/\tau_1/2) \]

\[ k_c = 1/(4\pi\sigma_0) = 8.98 \times 10^9 \text{ N}\text{m}^2\text{C}^{-2} \]
\[ R_H = 1.09678 \times 10^7 \text{ m}^{-1} \]

\[ \gamma = \frac{1}{(1-v^2/c^2)^{1/2}} = 1 + 0.5\beta^2 \text{ if } v \ll c \]
\[ \beta = v/c = [(\gamma^2-1)/\gamma]^{1/2} \]

\[ T = \frac{\gamma T_o}{L} \]
\[ \nu = \nu_o (1 + \beta)^{1/2} \text{ for } \beta \ll 1 \]
\[ \nu = c/\lambda \text{ (lect.)} = f(\text{book}) \]

\[ \check{\rho} = \gamma \mu \check{v} \text{ E} = \gamma m c^2 = K + m c^2 \]
\[ E^* = p^2 c^2 + m^2 c^4 \]

\[ E = pc = h\nu \quad \Delta \frac{\nu}{\nu} = \Delta \frac{T}{T} = -\frac{GM}{c^2} \left[ \frac{1 - \frac{1}{r_1}}{r_2} \right] \approx -\frac{gH}{c^2} \Delta \frac{\nu}{\nu} \approx -\frac{\Delta \lambda}{\lambda} \left( \text{if } \Delta \frac{\nu}{\nu} \ll 1 \right) \]
\[ R_{\text{Sch}} = \frac{2GM}{c^2} \]

\[ \lambda_{\text{max}} T = 2.898 \times 10^{-3} \text{ mK} \]
\[ I(\lambda,T) = \frac{2\pi c^2h}{\lambda^5} \frac{1}{e^{hc/\lambda T} - 1} \quad R(T) = \varepsilon \sigma T^4 \]

\[ h\nu = K_{\text{max}} + \phi \quad n\lambda = 2d \sin \theta \quad F_{\text{coul}} = k_c \frac{q_1 q_2}{r^2} \quad F_{\text{radiat}} = \frac{m v^2}{r} \quad a_o = \frac{4\pi \varepsilon_0 h^2}{m e^2} = 0.529 \text{ Å} \]

\[ \lambda = h/p \quad p = \hbar k \quad \Delta x \Delta p_x \geq 2\hbar \quad \Delta E \Delta t \geq \hbar/2 \]
\[ \nu_{ph} = \omega/k \quad \nu_{pr} = d\omega/dk \]
\[ e^{ikx} = \cos x + i \sin x \quad \cos x = \frac{1}{2}[e^{ix} + e^{-ix}] \quad \sin x = \frac{1}{2i}[e^{ix} - e^{-ix}] \]

\[ \sin 2t = 2 \sin t \cos t \quad \cos 2t = \cos^2 t - \sin^2 t = t \quad 2cos^2 t - 1 = 1 - 2 \sin^2 t \]
\[ \psi(x) = \frac{a_0}{2} + \sum_{n=1} \frac{\sum_{n=1} a_n \cos(\frac{2\pi n}{\ell} x) + \sum_{n=1} b_n \sin(\frac{2\pi n}{\ell} x)}{\ell^2} \text{ with } k_n = \frac{2\pi n}{\ell}, \text{ and} \]
\[ a_0 = \text{aver. of } \psi \text{ over } \ell, \quad a_n = \frac{2}{\ell} \int_0^\ell \psi(x') \cos(\frac{2\pi n}{\ell} x')dx' \quad b_n = \frac{2}{\ell} \int_0^\ell \psi(x') \sin(\frac{2\pi n}{\ell} x')dx' \]
\[ \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty c(k) e^{ikx} dx, \text{ with } c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \psi(x') e^{-ikx'} dx' \]
\[ \hat{H} \psi = i\hbar \frac{\partial \psi}{\partial t} \quad \hat{H} = \hat{K} + \hat{V} \quad \psi = \psi e^{\frac{iE}{\hbar}t} = \psi e^{-i\omega t} \quad \hat{H} \psi = E \psi \]
\[ \hat{p}_x = -i\hbar \frac{\partial}{\partial x} \quad \hat{K}_x = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \quad \Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \quad \hat{A}_{\psi,\alpha} = \hat{a} \psi_{\alpha} < A > = \int \psi^* \hat{A} \psi dx \]
\[ k = \sqrt{\frac{2m(E-V)}{\hbar^2}} \quad \psi \propto e^{\pm ikx} \quad \psi \propto \sin kx, \cos kx \quad \kappa = \alpha = \frac{1}{\delta} = \sqrt{\frac{2m(V-E)}{\hbar^2}} \quad \psi \propto e^{\pm x} \]

\[ \psi_n = \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{n \pi x}{L} \right) \quad E_n = \frac{n^2 \hbar^2}{2mL^2} \]

\[ \psi_n = H_n(x) e^{-ax^2/2} \quad \alpha = \sqrt{\frac{m \kappa}{\hbar^2}} \quad \omega = \sqrt{\frac{\kappa}{m}} \quad E_n = \left( n + \frac{1}{2} \right) \hbar \omega \]

---Tear off this sheet and begin exam---
Physics 9HE-Modern Physics
Midterm
February 3, 2011
(100 points total)

Name (printed)_______________________________________
Name (signature)_______________________________________
Student ID No._______________________________________

Are you interested in a Saturday tour of the Lawrence Berkeley National Laboratory? Yes_______No_______.
If yes, check dates that are OK: Feb 19______Feb 26______Mar 5______

[1] (50 Points) Four shorter questions on different topics.

(a) (10 points) A radar station sends out a continuous-wave emission at 3 cm wavelength and is observing raindrops in a hurricane heading toward it with velocities of 200 km/hr. What will be the beat frequency between the outgoing and reflected radar waves?

(b) (10 points) Light is emitted at a frequency of $7 \times 10^{14}$ Hz from a source sitting atop the Eiffel Tower in Paris, which is 321 m in height. What would be the fractional change in the frequency of this light as observed on the ground and in which direction would the frequency change (increase or decrease)?
(c) (15 points) The cosmic microwave background is actually Doppler shifted to appear to us as though it had a temperature of about 3 K, but in fact it was created at a temperature of about 3000 K. Use the formula for the maximum wavelength in the blackbody distribution curve, $\lambda_{\text{max}}T = 2.898 \times 10^{-3} \text{ m-K}$, to determine the speed at which this sheet of radiation is moving away from us. Just set up the relevant final equation, but you need not solve it.

\begin{align*}
\lambda_{\text{max}}T &= \text{constant} \\
\rightarrow \lambda_{\text{max}} &= \frac{\text{constant}}{T} \\
\rightarrow V &= \frac{cT}{\text{constant}} \\
\text{So } V(3000) &= \frac{3000}{8} = 1000; \quad V = \frac{V_0}{1-\beta} \\
\text{But } V &= V_0 \sqrt{\frac{1-\beta}{1+\beta}} \quad \Rightarrow \frac{1-\beta}{1+\beta} = \frac{1}{1000} \quad \Rightarrow 1-\beta = 2 \times 10^{-6}; \quad \beta = 0.999998
\end{align*}

So speed very near c!!! As expected for something that close to the edge of the universe in time, or rather very near its beginning, at least as we can observe it.

(d) (15 points) A monoenergetic beam of electrons produces a bright reflection spot from the face of a NaCl crystal with an incidence angle relative to the surface of $\theta = 30^\circ$. The spacing of the relevant planes is 0.28 nm. What is the accelerating voltage of the gun that produced these electrons? You may do this calculation non-relativistically.
Bragg reflections - wave like nature of electrons: Start with conservation of energy for electron motion through accelerating potential: Total $E_i = Total E_f \Rightarrow EPE_i + KE_i = EPE_f + KE_f$

where $EPE$ is the electrical potential energy and $KE$ is the kinetic energy. So $EPE_f = KE_i = 0$ and we have $EPE_i = KE_f \Rightarrow qV = \frac{1}{2}mv^2$

or (with $p = mv$) $V = \frac{1}{2m}p^2$. Now using the Bragg condition $n = 2dsin\theta$ and the de Broglie relationship ($\lambda = h/p$) we have $p = \frac{nh}{2dsin\theta}$ and substituted into $V$ we see: $V = \frac{1}{2mq} \left( \frac{nh}{2dsin\theta} \right)^2$

These are first order ($n = 1$) reflections and substituting gives $V \approx 14.7$ volts

\[ \text{(15 points)} \] Consider the Fourier series representation of the sawtooth waveform below
(a) (5 pts.) Why are all of the coefficients of cosines equal to zero. Use a symmetry argument.

The sawtooth wave is odd (anti-symmetric) about the origin, or stated another way the parity operator \( \hat{\Pi} \) acts on it as \( \hat{\Pi} f(x) = f(-x) = -f(x) \). The cosine is even (symmetric), with \( \cos(x) = \cos(-x) \), whereas the sine is odd (antisymmetric) with \( \sin(-x) = -\sin(x) \). So the sawtooth can be expanded completely using sine terms only.

(b) (5 pts.) Sketch the first two sine functions involved in this series on the single cycle below and give the integrals that would need to be evaluated in order to calculate the \( b_n \) values involved.

\[
\begin{align*}
\text{n = 1: } & \quad \sin \left( \frac{2\pi x}{L} \right) \\
\text{n = 2: } & \quad \sin \left( \frac{4\pi x}{L} \right)
\end{align*}
\]

\[
b_1 = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \psi(x) \sin \left( \frac{2\pi x}{L} \right) dx \\
b_2 = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \psi(x) \sin \left( \frac{4\pi x}{L} \right) dx
\]

(c) (5 pts.) Why is the \( b_2 \) coefficient negative? Again use a symmetry argument, and answer clearly with a sketch.

\[
b_2 \text{ is negative because the } b_2 \text{ sine term } \left( \sin \left( \frac{4\pi x}{L} \right) \right) \text{ begins positive (above } y = 0 \text{ line) and ends negative (below line). The original function begins negative and ends positive.}
\]
[3] (20 points) A one-dimensional particle in a rigid box with infinite potential in regions I and III and extending from 0 to L is described by the wave function \( \psi(x) = A \sin(3\pi x/L) \), with quantum number \( n = 3 \).

(a) (3 pts) Sketch the form of this wave function, as well as the probability of finding the particle at a given \( x \):

(b) (3 pts) Write down an expression for the normalization constant \( A \) in terms of a definite integral, but you need not evaluate the integral.

Note that the derivative is zero at the nodes of the Probability, since this is a square of the function.

(Continued on next page)
(b) (15 pts.) Write down an expression for the normalization constant $A$ in terms of a definite integral, but you need not evaluate the integral.

$$\int_0^L \psi_n^*(x) \psi_n(x) \, dx = 1 \quad \Rightarrow \quad A \int_0^L \sin^2 \left( \frac{3\pi x}{L} \right) \, dx = 1 \quad \Rightarrow \quad A = \sqrt{\frac{2}{L}}$$

(c) (15 pts.) Write down an expression for the average value of the position of the particle as obtained over many measurements, simplifying as far as you can without actually evaluating any integral. [Note: $d(\sin ax)/dx = a\cos ax$]

$$\psi(x) = A \sin \left( \frac{3\pi x}{L} \right)$$

$$\langle x \rangle = \int_0^L x \psi(x) \psi^*(x) \, dx = \int_0^L x \sin^2 \left( \frac{3\pi x}{L} \right) \, dx$$

(d) (15 pts.) Now use the symmetry of the problem and the functions involved to derive, without doing any mathematics, the answer to part (c).

*Expected value of position $\langle x \rangle$ is $L/2$, since it has equal probability of being $< L/2$ and being $> L/2$. Probability distribution function $(A^2 \sin^2 \left( \frac{3\pi x}{L} \right))$ is symmetric about $x = L/2$.

(e) (10 pts.) Now use the Uncertainty Principle and an approximate argument to estimate the minimum uncertainty in momentum of this particle in terms of $L$.

*Simple way: $\Delta p \Delta x \geq \hbar/2$ (uncertainty principle)

A source position of $x$ is "unknown" within the box, so $\Delta x \approx L$

Estimate $\Delta p \approx \hbar/2 \Delta x = \hbar/2L$
(f) (4 points) Finally, now let the walls of the box be of a finite height $V_0$, but still above the new energy $E_3$ of the particle, and write down the general forms of the trial wave functions in regions I, II, and III defined above.

$$
\begin{align*}
&\psi_i = A e^{\alpha x} + B e^{\beta x}, \\
&\psi_{II} = C \sin(kx + 3\cos kx), \\
&\psi_{III} = e^{-\alpha x} + F e^{-\beta x},
\end{align*}
$$

with: $\alpha = \sqrt{\frac{2m(V_0 - E_3)}{\hbar^2}}$, $k = -\sqrt{\frac{2mE_3}{\hbar^2}}$.

[4] (15 points) A certain bound quantum mechanical state $\Psi$ has been found to be a linear combination of two eigenfunctions $\phi_1$ and $\phi_2$ of an operator $\hat{A}$, such that $\hat{A}\phi_1 = 2\phi_1$ and $\hat{A}\phi_2 = 3\phi_2$, that are furthermore part of an orthogonal and normalized (orthonormal) set, such that

$$
\int_{-\infty}^{\infty} \phi_i^* \phi_j(x) \, dx = \delta_{ij}.
$$

The wave function is described by

$$
\Psi = c_1 \phi_1 + c_2 \phi_2,
$$

with $c_2 = \frac{c_1}{2}$. Determine the constants $c_1$ and $c_2$ and the average value of the observable $A$ over many measurements on this state.