Physics 9HE-Modern Physics
Sample Final Exam
(100 points total)

Miscellaneous data:

\[ c = 3.00 \times 10^8 \text{ m/s} \quad e = 1.60 \times 10^{-19} \text{ C} \quad 1 \text{ eV} = 1.60 \times 10^{-19} \text{ J} \quad 1 \text{ Å} = 10^{-10} \text{ m} \]

\[ M_{\text{Sun}} = 2 \times 10^{30} \text{ kg} \quad M_{\text{Earth}} = 5.98 \times 10^{24} \text{ kg} \quad r_{\text{Earth}} = 6.38 \times 10^6 \text{ m} \]

\[ m_e = 9.1094 \times 10^{-31} \text{ kg} = 0.5110 \text{ MeV/c}^2 \quad m_p = 1.6726 \times 10^{-27} \text{ kg} = 938.27 \text{ MeV/c}^2 \]

\[ m_n = 1.6749 \times 10^{-27} \text{ kg} = 939.57 \text{ MeV/c}^2 \quad m(1H) = 1.0078 \text{ u} \]

\[ 1 \text{ u} = 1.6605 \times 10^{-27} \text{ kg} = 931.49 \text{ MeV/c}^2 \]

\[ k_B = 1.38 \times 10^{-23} \text{ J-K}^{-1} \quad a_0 = 0.529 \text{ Å} \]

\[ N(t) = N_0 \exp(-t/\tau_{\text{O}}) = N_0 \exp(-0.693t/\tau_{1/2}) \]

\[ T = \gamma T_0 \quad L = L_0/\gamma \quad v = v_0 (\frac{1\pm\beta}{\Gamma(1\pm\beta)})^{1/2} \equiv v_0 [1 \pm \beta] \text{ for } \beta << 1 \quad v = c/\lambda \quad \text{v (lect.)} = f(\text{book}) \]

\[ x' = \gamma (x - vt) \quad y' = \gamma \quad z' = \gamma \quad t' = \gamma \left[ t - \left(\frac{v}{c^2}\right)x \right] \]

\[ u_x' = \frac{u_x - v}{1 - \left(\frac{v}{c}\right)^2 u_x} \quad u_y' = \frac{u_y}{\gamma} \quad u_z' = \frac{u_z}{\gamma} \]

\[ x^2 + y^2 + z^2 - c^2t^2 = \text{invariant} \]

\[ \Delta \lambda = \lambda' - \lambda = \frac{h}{mc} (1 - \cos \theta) = 2.43 \times 10^{-12} (1 - \cos \theta) \text{ (in m)} \]

\[ h = \frac{\hbar}{2\pi} \quad \pi \equiv \frac{1}{1 - (\frac{v}{c})^2} \quad \Delta \lambda = \frac{1}{\lambda} \quad \Delta \lambda = \frac{\hbar}{2\pi\varepsilon_0 K} \]

\[ F_{\text{coul}} = k_c \frac{q_1 q_2}{r^2} \quad F_{\text{radial}} = \frac{mv^2}{r} \quad E_n = -\frac{Z^2 e^2}{8\pi\varepsilon_0 a_n^2} = -\frac{13.62Z^2}{n^2} \text{ (eV)} \quad r_n = \frac{4\pi\varepsilon_0 \hbar^2}{m_e Z^2 n^2} = \frac{a_n n^2}{Z} \]

\[ a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{m_e \varepsilon_0^2} = 0.529 \text{ Å} \quad \varepsilon_n = \frac{nh}{m_e r_n} \quad \frac{1}{\lambda} = Z^2 R_H \left[ \frac{1}{n_l^2} \frac{1}{n_u^2} \right] \quad \mu_e = m_e \left[ \frac{M}{M + m_e} \right] \]

\[ \lambda = h/p \quad p = \hbar k \quad \Delta x \Delta p_x \geq \hbar/2 \quad \Delta E \Delta t \geq \hbar/2 \quad \nu_{\text{ph}} = \omega/\hbar \quad \nu_{\text{gr}} = d\omega/dk \]

\[ e^{ix} = \cos x + i \sin x \quad \cos x = \frac{1}{2} \left[ e^{ix} + e^{-ix} \right] \quad \sin x = \frac{1}{2i} \left[ e^{ix} - e^{-ix} \right] \]

\[ \sin 2t = 2 \sin t \cos t \quad \cos 2t = \cos^2 t - \sin^2 t = 2 \cos^2 t - 1 = 1 - 2 \sin^2 t \]
\[ \psi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{\ell} x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{\ell} x\right), \quad \text{with } k_n = \frac{2\pi n}{\ell}, \text{ and} \]

\[ a_n = \text{aver. of } \psi \text{ over } \ell, \quad a_n = \frac{1}{\ell} \int_0^\ell \psi(x') \cos\left(\frac{2\pi n}{\ell} x'\right) dx', \quad b_n = \frac{1}{\ell} \int_0^\ell \psi(x') \sin\left(\frac{2\pi n}{\ell} x'\right) dx' \]

\[ \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(k) e^{ikx} dk, \text{ with } c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x') e^{-ikx'} dx' \]

\[ \hat{H}\psi = i\hbar \frac{\partial\psi}{\partial t} \]

\[ \hat{H} = \hat{K} + V \]

\[ \psi = \psi e^{iEt/\hbar} = \psi e^{-i\omega t} \quad \hat{H}\psi = E\psi \]

\[ \hat{\rho}_x = -i\hbar \frac{\partial}{\partial x} \quad \hat{K}_x = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \quad \Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \quad \hat{A}_\alpha = a\psi_\alpha < A >= \int \psi^* \hat{A}\psi dx \]

\[ k = \sqrt{\frac{2m(E-V)}{\hbar^2}} \quad \psi \propto e^{i\kappa x} \quad \psi \propto \sin{k x}, \cos{k x} \quad \kappa = \alpha = \sqrt{\frac{2m(V-E)}{\hbar^2}} \quad \psi \propto e^{\pm kx} \]

\[ \psi_n = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right) \quad E_n = \frac{\pi^2 \hbar^2 n^2}{2ml^2} \]

\[ \psi_n = H_n(x) e^{-ax^2/2} \quad \alpha = \sqrt{\frac{m\kappa}{\hbar^2}} \quad \omega = \sqrt{\frac{\kappa}{m}} \quad E_n = \left(n + \frac{1}{2}\right)h\omega; \quad E_{rot} = \frac{\hbar^2\ell(\ell+1)}{2l} \quad I = \frac{mR^2}{2} \]

\[ T = \left[1 + \frac{V_0^2 \sin^2(\kappa L)}{4(E-V_0)}\right]^{1/2} \quad T = \left[1 + \frac{V_0^2 \sinh^2(\kappa L)}{4(E-V_0)}\right]^{1/2} \approx 16 \frac{E}{V_0} \left[1 - \frac{E}{V_0}\right] e^{-2\kappa L} \text{ (when } \kappa L \gg 1) \]

\[ T_{FE} \approx \exp\left[-\frac{4\phi^{3/2}\sqrt{2m_e}}{3e\hbar dx}\right] \quad I_{STM} \propto e^{-2KL} \quad \lambda = f_{cell}T_a; \quad T_a = \exp\left[-4\pi Z\sqrt{\frac{0.0993 \text{ MeV}}{E_a(\text{MeV})}} + 8\sqrt{\frac{Z\text{R}_{\text{nuc}}(m)}{7.3 \times 10^{-15}}}\right] \]

\[ x = r \sin\theta \cos\phi \quad y = r \sin\theta \sin\phi \quad z = r \cos\theta \quad dV = r^2 dr \sin\theta d\theta d\phi \]

\[ \psi_{\text{norm}}(r,\theta,\phi) = R_{\text{nf}}(r) \Theta_{\text{nf}}(\theta) \Phi_{\text{nf}}(\phi) = R_{\text{nf}}(r) Y_{\text{nf}}(\theta,\phi) \quad P_{\text{nf}}(r) = r^2 R_{\text{nf}}^2(r) \]

\[ \mu = iA \quad E = -\mu_B \quad \mu_B = -\frac{e}{2m} \quad \mu_0 = -\frac{e}{m} = -2\mu_0 \quad \mu_0 = \frac{e\hbar}{2m} = 5.058 \times 10^{-27} \text{ J/Tesla} \]

\[ \psi_j^{MO}(\vec{r}) = \sum_{\text{Atoms } i} c_{ij} \Psi_{Al}(\vec{r}) \]

\[ B\left(\frac{1}{2}\right)X = [Nm_A + Zm(\frac{1}{2})]c^2 \approx a_A - a_{A^2/3} - \frac{3}{5} \frac{Z(Z-1)e^2}{4\pi\varepsilon_0 R_{\text{nuc}}} - a_s \frac{(N-Z)^2}{A} + \delta \]

\[ R_{\text{nuc}} \approx (1.2 \times 10^{-15}) m \quad A^{1/3} \quad Q = [M_{\text{initial}} - M_{\text{final}}]c^2 \]

---Tear off this sheet and begin exam---
The free neutron at rest decays into a proton and an electron with an exponential lifetime of $\tau_0 = 920$ s. Now consider a beam of neutrons with a kinetic energy of 100 MeV travelling along the +x direction relative to an observer in the laboratory.

(a) At what speed will the neutrons in the beam be moving?

\[
\gamma \frac{mc^2}{E} = \frac{\gamma v c}{E} = \frac{1 - \frac{v^2}{c^2}}{\frac{1}{\gamma} - \frac{v^2}{c^2}} = 1 - \left(\frac{\gamma v}{c}\right)^2
\]

\[
v = 0.42c = 1.26 \times 10^8 \text{ m/s}
\]

(b) What would the observer in the laboratory measure for the lifetime of the neutrons in the beam?

\[
\left[\frac{\tau}{\gamma}\right]_{\text{LAB}} = \frac{\tau}{\gamma} = 1.107(920 \text{ s}) = 1018.5
\]
(a) State three types of experimental observations that challenged classical physics around 1900.

Any three of:

Blackbody radiation
Electromagnetism and the origin of magnetic fields, which did not transform from one coordinate system to another
Line spectra of hydrogen and other atoms
The photoelectric effect
Compton scattering

(b) State three experimentally verified consequences of Special Relativity.

Any three of:
Time dilation, e.g. in planes circling earth
Length contraction, e.g. in muon viewing mountain
Relativistic momentum: effective mass increase of moving object as viewed in fixed frame
Relativistic energy: \( E = mc^2 \)
Special version of Doppler effect
B fields as a relativistic manifestation of E fields

(c) So-called L x-rays are emitted from a copper atom in which an initial 2p vacancy is created. What transitions from the \( n = 3 \) shell are permitted in generating these x-rays and why?

Dipole selection rules are: \( \Delta \ell = \pm 1 \) and \( \Delta m_\ell = 0, \pm 1 \), so the allowed transitions are from 3d with \( \Delta \ell = 1 \rightarrow -2 = -1 \) and 3s with \( \Delta \ell = 1 \rightarrow 0 = +1 \), and this would get full credit. In more detail using the other selection rule, we would have:

\[
\begin{align*}
3d: \ell = 2, m_\ell &= -2, -1, 0, +1, +2 \\
3s: \ell = 0, m_\ell &= 0 \\
2p: \ell = 1, m_\ell &= -1, 0, +1
\end{align*}
\]
(d) A red laser is used to produce a perfect sinusoidal traveling light wave of frequency $\nu_0 = 10^{14}$ Hz. However, by special means, the wave is abruptly cut off at each end so that it has a finite length in time of $10^{-13}$ s. If this finite wave is now described in a Fourier representation, what would be the approximate range of frequencies $\Delta \nu$ involved?

\[
\Delta \nu = \frac{1}{\nu_0} = \frac{1}{10^{14}} \text{ Hz}
\]

(e) Define degeneracy and give one example of it from the systems we have studied.

\[
\text{Degenerate wave functions are those which are non-identical and mathematically independent, but still correspond to the same energy. I.e.,} \quad \psi_1 \neq \psi_2, \quad \text{but} \quad \hat{H} \psi_1 = E \psi_1, \quad \hat{H} \psi_2 = E \psi_2 \\
\text{Any one OK, but we have seen following examples:} \\
\bullet \text{1D free particle:} \quad E = \frac{\hbar^2}{2m} \left( k \right)^2 \text{ where } k \text{ is kinetic energy} \\
\bullet \text{3D particle in a rigid box:} \quad E = E_x + E_y + E_z, \quad n_x = 1, 2, 3, \ldots \quad n_y = 1, 2, 3, \ldots \quad n_z = 1, 2, 3, \ldots \\
\bullet \text{Hydrogenic atom:} \quad E_n = \frac{n(n+1)}{2} \frac{\hbar^2}{m} \text{ where } n \text{ is allowed for } l \\
\bullet \text{Many-electron atoms:} \quad E_n \ldots \ldots \text{ are allowed for that } l
\]

(f) The position of an electron along the x direction is measured with an uncertainty of 1 Å. What can you say about the uncertainty in its x momentum? What, if anything, can you say about the uncertainty in its y momentum?

\[
\Delta x \Delta p_x \geq \frac{\hbar}{2} \quad \rightarrow \quad \Delta p_x \geq \frac{\hbar}{2 \Delta x} = \frac{1.05 \times 10^{-34} \text{ J s}}{2 \times 10^{-10} \text{ m}} = 5.28 \times 10^{-25} \text{ kg m/s} \\
\Delta p_y \text{ is not involved with x coordinate measurement, so can't say anything}
\]

(f) In the oxygen molecule pointing along the x direction, one of the bonding wave functions can be made up of oxygen 2px functions on the two atoms. Indicate the
equation for this wave function, and sketch it, using a 3D contour of equal probability density.

\[
\psi_{\text{Molecular Orbital}} \propto \phi_{O_2 p_x \text{ on atom 1}} - \phi_{O_2 p_x \text{ on atom 2}}
\]

(Simple sign OK too)
[1.3] (15 points) Consider a hydrogenic atom consisting of a single electron around a uranium nucleus (Z = 92).

(a) What will be the energy in eV and the radius in Å of the first Bohr orbit for this atom?

\[
E_n = \frac{-(13.6)Z^2}{n^2} \Rightarrow E_1 = \frac{-(13.6)(92)}{1^2} = -115,910 \text{ eV}
\]

\[
r_n = \frac{n^2a_0}{\varepsilon} \Rightarrow r_1 = \frac{1^20.529\text{ Å}}{92} = 0.00575\text{ Å} = 5.75 \times 10^{-13}\text{ m}
\]

(b) Balance coulombic force and acceleration and use the result of part (a) to derive the speed of the electron in the first Bohr orbit around this atom, using a non-relativistic approach. Based on your answer, is a non-relativistic model adequate?

\[
F = ma = F_{\text{coul}} \Rightarrow \frac{mv^2}{r} = \frac{k_eZe^2}{r^2} \Rightarrow v = \sqrt{\frac{k_eZe^2}{mr}}
\]

\[
v = \sqrt{\frac{(8.98 \times 10^9)(92)(1.602 \times 10^{-19})^2}{(9.11 \times 10^{-31})(5.75 \times 10^{-13})}} \approx 2.011 \times 10^6\text{ m/s} \approx 0.001\text{ c}
\]

(c) Now consider this hydrogenic atom initially in an n = 10 excited state, from which it deexcites to the n = 1 state by emitting electromagnetic radiation. What would be the weight change of the atom due to emitting this radiation, including its sign (increase or decrease)?

\[
\Delta m = \frac{\Delta E}{c^2} = \frac{E_{10} - E_1}{c^2} = (13.6\text{ eV})(92)^2 \left(\frac{1}{7^2} - \frac{1}{10^2}\right)
\]

\[
= 20,284 \times 10^{-35}\text{ kg} = 2.03 \times 10^{-31}\text{ kg}, \text{ about 1/5 the mass of an electron!}
\]

Energy lost, so mass decreases: \(\Delta m = \text{negative}\)
[1.4] (10 points) Consider an electron trapped inside the potential well shown below, at an energy of 200 eV, at the position indicated:

(a) Qualitatively sketch on the diagram the form of the wavefunction in each of the regions 1-4 indicated, being careful to show the relative wavelengths and the relative amplitudes in each region. (That is, in which region of the box will the particle be most likely to be found?)

From (c) below, the wavelength is pretty small compared to this well's dimensions, so there are something like 50/0.868 = 57 cycles in each of regions 2 and 3. Too hard to draw this, but I have indicated relative heights of $\psi$ such that $|\psi|^2$ is inversely proportional to the particle velocity in the region (a correspondence principle type of argument), which is in turn proportional to the square root of kinetic energy. So $|\psi_2|^2/|\psi_3|^2 \approx [KE_3/KE_2]^{1/2} = [100/200]^{1/2} = 0.707$ and finally $\psi_2/\psi_3 \approx 0.840$. Particle is more likely to be found in region 3.

Full credit here was simply showing a smaller wavelength and amplitude in region 2.

(b) Indicate the classical turning points appropriate to this state on the diagram as well.

See diagram. Two of them.
(c) For the energy values shown on the diagram, what is the wavelength of the particle in region 2?

In general, \( \lambda = h/p \), and in non-relativistic limit energy of particle in region 2 = \( E_2 = KE \) = kinetic energy = \( p^2/2m \), so \( p = \sqrt{2mE} \) and finally \( \lambda = h/\sqrt{2mE} \).

With \( E = 200 \text{ eV} = 200(1.60 \times 10^{-19} \text{ J/eV}) = 3.20 \times 10^{-17} \text{ J} \), we thus have finally, in SI units:

\[
\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.63 \times 10^{-34} \text{ J-s}}{\sqrt{2(9.11 \times 10^{-31} \text{ kg})(3.20 \times 10^{-17} \text{ J})}} \\
= \frac{6.63 \times 10^{-34}}{7.63 \times 10^{-24}} = 0.868 \times 10^{-10} \text{ m} = 0.868 \text{ Å}
\]

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[1.5] (15 Points) Consider an electron incident from the left side of the infinitely long potential step at \( x = 0 \) shown below. The particle energy \( E = 10 \text{ eV} \) and the step has a height of \( V_0 = 7 \text{ eV} \).

![Diagram of electron incident on potential step](image)

(a) What is the relevant time-independent Schroedinger equation to the left of the step (region I) and to the right of the step (region II)? You need not solve these equations.
(b) What would be the time-dependent form of the wave function for this problem in both regions, including both transmitted and reflected components and taking advantage of the known general solutions to this type of problem?
(d) Now set up the boundary conditions needed to solve this problem, but you need not go beyond this.

[1.6] (15 Points) Two of the 3d hydrogen-atom wavefunctions are given by:

\[ \psi_{3,2,\pm 2}(r, \theta, \phi) = C r^2 e^{-\frac{r}{\kappa}} \sin^2 \theta e^{\pm 2i\phi} \]

(a) Indicate how the probability density associated with these wavefunctions would be calculated, and how the normalization constant C would be determined from this density, going as far as you can without evaluating any non-trivial integrals.

(b) Sketch and describe in words in an unambiguous way the three-dimensional probability density of these two hydrogenic wave functions, using either a 3D contour of equal probability or a plot in which greater darkness means a higher probability of finding the electron at a given position.
(c) Write down three eigenfunction relations satisfied by these wave functions, including a specification of the eigenvalues involved and the physical meaning of the eigenvalue.

\[
\hat{\mathbf{L}}^2 \psi_{3,2,\pm \pm} = \mathbf{L}^2 \psi_{3,2,\pm \pm} = \hbar^2 (2+1) \psi_{3,2,\pm \pm}
\]

\[
\hat{\mathbf{L}}_z \psi_{3,2,\pm \pm} = \pm \hbar \psi_{3,2,\pm \pm}
\]

\[
\mathbf{H} \psi_{3,2,\pm \pm} = E_{3d_{\pm \pm}} \psi_{3,2,\pm \pm}
\]

(d) Write down the time-dependent form of these two wave functions.

_Just add the usual complex exponential as:_

\[
\psi_{3,2,\pm \pm}(r, \theta, \phi) = \psi_{3d_{\pm \pm}} = \mathbf{Cr}^2 e^{\frac{r}{3\omega}} \sin^2 \theta e^{i(2\omega t - \hbar \omega)}
\]

where \( \omega = E_3 / \hbar = -(13.6 \text{ eV}) \frac{1}{3^2 \hbar} = -\frac{1.51 \text{ eV}}{\hbar} \)

[1.7] (10 Points) Consider the nuclide \(^{49}_{24}\text{Cr}_x\) of chromium, with atomic mass of 48.951341 u.

(a) What is \(x\) here, and what is the binding energy of this nuclide per nucleon?

\[
E_b = \frac{\text{ binding energy}}{\text{ nucleon}} = \frac{E_b}{48.951341} = \frac{1}{49} \left[ 24 \times m_p c^2 + 25 \times m_n c^2 \right] = \frac{1}{49} \left[ 24 \times (938,246) + 25 \times (938,527) \right] - \frac{48.951341}{938.527} \frac{1440}{1} \frac{1}{938.527} = 0.36 \text{ MeV per nucleon}
\]

(b) This nuclide decays by positron emission to form a nuclide of vanadium = V. Write down the overall reaction, including the new nuclide that would be formed.

_**Positron emission reduces the atomic no. Z by one, and overall reaction is:**_

\[
^{49}_{24}\text{Cr}_{25} \rightarrow ^{49}_{23}\text{V}_{26} + \text{e}^+ + \nu_e
\]
OK without neutrino stated.

(c) Which one of the four fundamental interactions is responsible the force between quarks and what is the particle mediating this force?

The strong interaction is responsible for quark interaction and the mediating particle is the gluon.

---End of examination---