1. (30 points) You are going to calculate the properties of an ultrarelativistic proton, with mass $m$, as it passes first through an accelerating voltage, followed by a velocity selector. Assume that the initial velocity of the proton is zero. See the diagram of the arrangement in the lab frame below.

(a) (5 points) The proton will gain energy from the accelerating voltage $V$ along $\hat{x}$. Find its velocity after it passes through the electric field caused by the accelerating voltage.

(b) (5 points) Next, the proton passes through velocity selector, which is a region of crossed uniform electric and magnetic fields. The fields are both perpendicular to the particle’s velocity and to each other, i.e., $\vec{v} = v \hat{x}$, $\vec{E} = E \hat{y}$, $\vec{B} = B \hat{z}$ (see diagram). Derive an expression for the magnitude of the velocity $v$, in terms of the magnitudes of the electric and magnetic fields, $E$ and $B$, if the proton’s velocity $v$ is unchanged as it goes through this region, i.e., the proton successfully goes through the entrance slit to the velocity selector and leaves through the exit slit, with both slits along the x-axis, as shown.

(c) (2 points) What is the work done by the magnetic field on the proton?

(d) (10 points) Now consider the velocity selector from the rest frame of the proton. Since the proton went through the slits in the lab frame, the slits must go around the proton in the proton’s rest frame. Therefore, there must be no force acting on the proton in its rest frame. Show this by using the Lorentz transformation equations (below) to find the $\vec{E}'$ and $\vec{B}'$ fields in the proton’s rest frame. From the fields in the proton’s rest frame, now explicitly calculate the force on the proton in its rest frame and show that it is indeed zero.

(e) (8 points) For the velocity selector, demonstrate explicitly by using the values for the fields in both frames that $E^2 - B^2$ and $\vec{E} \cdot \vec{B}$ are both relativistic invariants.

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**Lorentz transformation equations for fields, with primed quantities measured in frame $F'$ moving in +x direction with speed $v$ as seen from F.**

$$
E'_x = E_x \\
E'_y = \gamma (E_y - \beta B_z) \\
E'_z = \gamma (E_z + \beta B_y) \\
B'_x = B_x \\
B'_y = \gamma (B_y + \beta E_z) \\
B'_z = \gamma (B_z - \beta E_y)
$$

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Diagram below shows cross sectional view of apparatus described in the problem
2. (10 pts) (a) Begin with Maxwell’s equations with no sources in vacuum. Using the vector identity given below, derive the wave equation with differential operators in both space and time for either the electric field \( \vec{E}(\vec{x}, t) \) or the magnetic field \( \vec{B}(\vec{x}, t) \). What is the speed of the wave?

(b) Are electromagnetic waves transverse or longitudinal? Give two examples of electromagnetic waves.

\[
\vec{\nabla}_x (\vec{\nabla}_x \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2 \vec{a}
\]

3. (10 points) (a) Use Gauss’s Law to calculate the \( \vec{E} \) field between two concentric spherical conducting shells, the inner one with radius \( a \) and the outer one with radius \( b \). A charge of \(+Q\) is on the inner shell and a charge of \(-Q\) on the outer shell.

(b) Derive the formula for the capacitance of the configuration.

(c) Find the energy stored in the capacitor.

4. (25 points) Consider a cylindrical charge distribution \( \rho \) of radius \( a \) and length \( L \) lying along the \( z \) axis. Assume that \( L \gg a \), and ignore any end effects. For \( r < a \), \( \rho(r) = \rho_o r^3 \), where \( \rho_o \) is a constant. The cylindrical coordinates are \( (r, \theta, z) \).

(a) Find the electric field everywhere, both inside and outside the charge distribution.

(b) Find the potential everywhere, both inside and outside the charge distribution, assuming that \( \varphi(r = 0) = 0 \).

(c) Show that the potential satisfies Poisson’s equation inside the cylinder.

In cylindrical coordinates, \( \nabla^2 \varphi(r, \theta, z) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} \)

5. (10 points) Two wires are carrying current as shown below. \( I_1 = 10^9 \) esu/sec. What must be the current \( I_2 \) if the magnetic field at point \( P \) is to be zero? \( x = 2 \) cm, \( d = 8 \) cm.

![Diagram of two wires and a point P](image)

6. (20 points) Consider a solenoid, with 2000 turns, length 50 cm, radius 2 cm, and current of \( 3 \times 10^{10} \) esu/sec.

(a) Use Ampere’s Law to derive the formula for the \( \vec{B} \) field inside a solenoid, and then calculate the value of the \( \vec{B} \) field for the solenoid with the given parameters. (Neglect any end effects, i.e., pretend the solenoid is infinitely long.)

(b) Derive the formula for the self-inductance of the solenoid and calculate its value.

(c) Write the formula for energy in the magnetic field of the solenoid and show that it is indeed equal to \( \frac{1}{2} LI^2 \) for this case. Find the numerical value of the energy when the given current is flowing in the solenoid.
7. (20 points) Consider an RLC series circuit, with \( R = 1\text{k}\Omega \), \( L = 1.0\text{H} \), and \( C = 1\mu\text{F} \). The circuit is driven by an ideal ac current source \( I_o \cos \omega t \), where \( I_o = 5\text{A} \) and \( \omega = 200\text{ sec}^{-1} \).

(a) Find the magnitude and the phase of the voltage across the whole circuit, i.e., find \( V \) and \( \phi \) for the resulting voltage \( V \cos(\omega t + \phi) \).

(b) Find the voltage and phase across the inductor alone.

(c) Find the resonant frequency \( \omega_R \) of the circuit. What is the magnitude of the equivalent impedance for the RLC series combination at that frequency? What are the magnitude and phase of \( V \) at that frequency?

8. (25 points) Consider a U-shaped conductor with right angles and negligible resistance, as shown. The conductor is placed into a region of uniform, magnetic field \( \vec{B} = -B_0 \hat{z} \), where \( B_0 = \text{constant} \) and the \( -\hat{z} \) direction points into the paper (see x’s on diagram representing the direction of the magnetic field.) A metal bar with mass \( m \) and resistance \( R \) is placed across the U-shaped conductor and makes good electrical contact with it at the contact points as it moves frictionlessly. The length of the bar between the contact points is \( L \).

(a) Initially, an external force is applied to the bar to move it to the right with constant velocity \( v_o \). Find the EMF (i.e., voltage) induced in the loop and the power dissipated in the bar. Determine the direction of the current and explain how you figured that out.

(b) At time \( t_o \), the external force acting on the bar is removed. Find the speed of the bar as a function of time for \( t > t_o \). What is the power dissipated in the bar as a function of time? Find the total energy dissipated in the bar after time \( t_o \) and show that it is equal to \( \frac{1}{2} m v_o^2 \).