Chapter 7

1. Starting with Equation (7.7), let the electron move in a circle of radius \( a \) in the \( xy \)-plane, so \( \sin \theta = 1 \). With both \( r \) and \( \theta \) constant, \( R \) and \( f \) are also constant. Let \( R = f = 1 \). Then \( g = \psi \) and the derivatives of \( R \) and \( f \) are zero. With this Equation (7.7) reduces to

\[
-\frac{2\mu}{\hbar^2}a^2 (E - V) = \frac{1}{\psi} \frac{d^2 \psi}{d\phi^2}
\]

3. Assuming a trial solution \( g = Ae^{ik\phi} \) (which is easily verified by direct substitution), and using the boundary condition \( g(0) = g(2\pi) \), we find

\[
Ae^0 = Ae^{2\pi k}
\]

which is only true if \( k \) is an integer.

5. Letting the constants in the front of \( R \) be called \( A \) we have

\[
R = A \left( 2 - \frac{r}{a_0} \right) e^{-r/2a_0}
\]

\[
\frac{dR}{dr} = A \left( \frac{2}{a_0} + \frac{r}{2a_0^2} \right) e^{-r/2a_0}
\]

\[
\frac{d^2 R}{dr^2} = A \left( \frac{3}{2a_0^2} - \frac{1}{4a_0^3} \right) e^{-r/2a_0}
\]

Substituting these into Equation (7.13) we have

\[
\left( -\frac{1}{4a_0^3} \right) - \frac{2\mu E}{a_0^2 \hbar^2} \right) + \frac{5}{2a_0^2} + \frac{4\mu E}{4\pi \varepsilon_0 a_0 \hbar^2} - \frac{2\mu e^2}{4\pi \varepsilon_0 a_0 \hbar^2} + \frac{4}{a_0} + \frac{4\mu e^2}{4\pi \varepsilon_0 a_0 \hbar^2} \right) \frac{1}{r} = 0
\]

To satisfy the equation, each of the expressions in parentheses must equal zero. From the \( 1/r \) term we find

\[
a_0 = \frac{4\pi \varepsilon_0 \hbar^2}{\mu e^2}
\]

which is correct. From the \( r \) term we get

\[
E = -\frac{\hbar^2}{8\mu a_0^2} = \frac{E_0}{4}
\]

which is consistent with the Bohr result. The other expression in parentheses also leads directly to \( E = -E_0/4 \), so the solution is verified.

8. Do the triple integral over all space

\[
\iiint \psi^* \psi \, dV = \frac{1}{\pi a_0^3} \int_0^{2\pi} \int_0^\pi \int_0^\infty r^2 \sin \theta \, e^{-2r/\alpha_0} \, dr \, d\theta \, d\phi
\]

The \( \phi \) integral yields \( 2\pi \), and the \( \theta \) integral yields \( 2 \). This leaves

\[
\iiint \psi^* \psi \, dV = \frac{4\pi}{\pi a_0^3} \int_0^\infty r^2 \, e^{-2r/\alpha_0} \, dr = \frac{4}{a_0^3} \left( \frac{2}{2/\alpha_0} \right)^3 = 1
\]

as required.

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\[
\psi_{310} = R_{31} Y_{10} = \frac{1}{8} \sqrt{\frac{2}{\pi}} a_0^{-3/2} \left( 6 - \frac{r}{a_0} \right) \left( \frac{r}{a_0} \right) e^{-r/\alpha_0} \cos \theta
\]

\[
\psi_{31\pm1} = R_{31} Y_{1\pm1} = \frac{1}{8\sqrt{\pi}} a_0^{-3/2} \left( 6 - \frac{r}{a_0} \right) \left( \frac{r}{a_0} \right) e^{-r/\alpha_0} \sin \theta e^{\pm\phi}
\]
13. As in Example 7.4 the degeneracy is \( n^2 - 36 \).

* 15. 

\[ \cos \theta = \frac{L_z}{L} = \frac{m_l}{\sqrt{\ell(\ell+1)}} \]

For this extreme case we could have \( \ell = m_l \) so

\[ \cos (3^\circ) = \frac{\ell}{\sqrt{\ell(\ell+1)}} \quad \cos^2 (3^\circ) = \frac{\ell^2}{\ell(\ell+1)} - \frac{\ell^2}{\ell^2 + \ell} \]

Rearranging we find

\[ \ell = \left( \frac{1}{\cos^2 (3^\circ)} - 1 \right)^{-1} = 364.1 \]

and we have to round up in order to get within 3°, so \( \ell = 365 \).

16. There is one possible \( m_l \) value for \( \ell = 0 \), three values of \( m_l \) for \( \ell = 1 \), five values of \( m_l \) for \( \ell = 2 \), and so on, so that the degeneracy of the \( n \)th level is

\[ 1 + 3 + 5 + \ldots = n^2 \]

19. With \( \ell = 1 \) we have \( m_l = 0, \pm 1 \) and \( L_z = m_l \hbar = 0, \pm \hbar \).

20. The maximum difference is between the \( m_l = -2 \) and \( m_l = +2 \) levels, so \( \Delta m_l = 4 \). Then

\[ \Delta V = \mu_B (\Delta m_l) B = (5.788 \times 10^{-5} \text{ eV/T}) (4) (2.5 \text{ T}) = 5.79 \times 10^{-4} \text{ eV} \]

29. For the \( 5f \) state \( n = 5 \) and \( \ell = 3 \). The possible \( m_l \) values are \( 0, \pm 1, \pm 2 \), and \( \pm 3 \) with \( m_s = \pm 1/2 \) for each possible \( m_l \) value. The degeneracy of the \( 5f \) state is then (with 2 spin states per \( m_l \)) equal to \( 2(7) = 14 \).

32. The spin degeneracy is 2 and the \( n^2 \) is shown in Problem 16.

* 35. We must find the maxima and minima of the following function.

\[ P(r) = r^2 |R(r)|^2 = A^2 e^{-r/a_0} \left( 2 - \frac{r}{a_0} \right)^2 \]

To find the extrema set \( \frac{dP}{dr} = 0; \)

\[ 0 = -\frac{1}{a_0} \left( 4r^2 - \frac{4r^3}{a_0} + \frac{r^4}{a_0^2} \right) e^{-r/a_0} + \left( 8r - \frac{12r^2}{a_0} + \frac{4r^3}{a_0^2} \right) e^{-r/a_0} \]

\[ 0 = \frac{r^3}{a_0^2} + \frac{8r^2}{a_0} - \frac{16r}{a_0} + 8 \]

Letting \( x = \frac{r}{a_0} \) the equation above can be factored into \( (x^2 - 2)(x^2 - 6x + 4) = 0 \). From the first factor we get \( x = 2 \) (or \( r = 2a_0 \)), which from Figure 7.12 we can see is a minimum. The second parenthesis gives a quadratic equation with solutions \( x = 3 \pm \sqrt{5} \), so \( r = (3 \pm \sqrt{5})a_0 \). These are both maxima.
38. The radial probability distribution for the ground state is

\[ P(r) = r^2 |R(r)|^2 = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \]

With \( r \ll a_0 \) throughout this interval we can say \( e^{-2r/a_0} \approx 1 \). Therefore the probability of being inside a radius \( 10^{-15} \) m is

\[ \int_0^{10^{-15}} P(r) \, dr \approx \frac{4}{a_0^3} \int_0^{10^{-15}} r^2 \, dr = \frac{4r^3}{3a_0^3} \bigg|_0^{10^{-15}} = 9.0 \times 10^{-15} \]

46. a) The only change in Equation (7.3) is in the potential energy, with

\[ V = -\frac{Z\alpha^2}{4\pi\epsilon_0 r} \]

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{2\mu}{\hbar^2} \left( E + \frac{Z\alpha^2}{4\pi\epsilon_0 r} \right) \psi = 0 \]

b) Because \( V \) occurs only in the radial part, there is no change in the separation of variables.

c) Yes, from Equation (7.10) it is clear that the radial wave functions will change.

d) No, there is no change in the \( \theta \) or \( \phi \) dependence.

**Special problem**

Neglecting the radial part of the wave function,

\[ 3d_{+1} + 3d_{-1} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin \theta \, e^{i\phi} + \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin \theta \, e^{-i\phi} = \]

\[ = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin \theta \left( e^{i\phi} + e^{-i\phi} \right) = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \phi \]

\[ = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \cos \phi \]

Since in polar coordinates

\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]

\[ \Rightarrow 3d_{+1} + 3d_{-1} = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \left( x^2 + y^2 \right) \sim x^2 + y^2 \]

\[ 3d_{+2} - 3d_{-2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin \theta \left( e^{i\phi} - e^{-i\phi} \right) = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \sin \phi \]

\[ = \frac{1}{2} \sqrt{\frac{15}{2\pi}} xy \sim xy \]