[1] (30 points) Consider the Fourier series representation of the sawtooth waveform below

(a) (10 pts.) Why are all of the coefficients of cosines equal to zero. Use a symmetry argument.

Sawtooth wave is odd (anti-symmetric) about origin \( f(-x) = -f(x) \)

Cosine function is even (symmetric): \( \cos(-x) = \cos x \). So sawtooth wave can be expanded completely with sine terms (odd)
(b) (10 pts.) Sketch the first two sine functions involved in this series on the single cycle below and give the integrals that would need to be evaluated in order to calculate the \( b_n \) values involved.

\[ n = 1: \quad \sin \left( \frac{2\pi x}{L} \right) \]

\[ n = 2: \quad \sin \left( \frac{4\pi x}{L} \right) \]

\[ b_1 = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \psi(x) \sin \left( \frac{2\pi x}{L} \right) dx \]

\[ b_2 = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \psi(x) \sin \left( \frac{4\pi x}{L} \right) dx \]

(c) (10 pts.) Why is the \( b_2 \) coefficient negative? Again use a symmetry argument.

\( b_2 \) is negative because the \( b_2 \) sine term \( \sin \left( \frac{4\pi x}{L} \right) \)

begins positive (above \( y = 0 \) line) and ends negative (below line).

The original function begins negative and ends positive.

[2] (70 points) A one-dimensional particle in a rigid box extending from 0 to \( L \) is described by the wave function \( \psi(x) = A \sin(3\pi x/L) \), with quantum no. \( n = 3 \).

(a) (15 pts) Sketch the form of this wave function, as well as the probability of finding the particle at a given \( x \):
(b) (15 pts.) Write down an expression for the normalization constant \( A \) in terms of a definite integral, but you need not evaluate the integral.

\[
\int_0^L \psi_n^*(x) \psi_n(x) \, dx = 1
\]

\[
A^2 \int_0^L \sin^2\left(\frac{3\pi x}{L}\right) \, dx = 1
\]

\[
\Rightarrow \quad A = \sqrt{\frac{2}{L}}
\]

(c) (15 pts.) Write down an expression for the average value of the position of the particle as obtained over many measurements, simplifying as far as you can without actually evaluating any integral. (Note: \( \frac{d}{dx}(\sin ax) = a\cos x \))

\[
\psi(x) = A \sin\left(\frac{3\pi x}{L}\right)
\]

\[
\langle x \rangle = \int_0^L x \psi(x) \, dx = A^2 \int_0^L x \sin^2\left(\frac{3\pi x}{L}\right) \, dx
\]

(d) (15 pts.) Now use the symmetry of the problem and the functions involved to derive, without doing any mathematics, the answer to part (c).

- Expected value of position \( \langle x \rangle \) is \( 4/2 \), since it has equal probability of being \( <4/2 \) and \( >4/2 \).
- Probability distribution function \( A^2 \sin^2\left(\frac{3\pi x}{L}\right) \) is symmetric about \( x = 4/2 \).

(e) (10 pts.) Now use the Uncertainty Principle and an approximate argument to estimate the minimum uncertainty in momentum of this particle in terms of \( L \).

Simple way: \( \Delta p \Delta x \geq \frac{\hbar}{2} \) (uncertainty principle)

Assume position of \( x \) is "unknown" within the box, so \( \Delta x = L \)

Estimate: \( \Delta p \approx \frac{\hbar}{2\Delta x} = \frac{\hbar}{2L} \)
[1] (30 points) Consider the Fourier series representation of the sawtooth waveform below.

(a) (10 pts.) Why are all of the coefficients of cosines equal to zero. Use a symmetry argument.

Sawtooth wave is odd (anti-symmetric) about origin: \( f(-x) = -f(x) \)

Cosine function is even (symmetric): \( \cos(-x) = \cos(x) \). So sawtooth wave can be expanded completely with sine terms (odd).
(b) (10 pts.) Sketch the second two sine functions involved in this series on the single cycle below and give the integrals that would need to be evaluated in order to calculate the $b_n$ values involved.

$$n = 2: \quad \sin \left( \frac{4\pi x}{L} \right)$$

$$n = 3: \quad \sin \left( \frac{6\pi x}{L} \right)$$

\[ b_2 = \frac{2}{L} \int_{-L/2}^{L/2} \psi(x) \sin \left( \frac{4\pi x}{L} \right) dx \]

\[ b_3 = \frac{2}{L} \int_{-L/2}^{L/2} \psi(x) \sin \left( \frac{6\pi x}{L} \right) dx \]

(c) (10 pts.) Why is the $b_2$ coefficient negative? Again use a symmetry argument.

$b_2$ is negative because $b_2$ sine term \( \sin \left( \frac{4\pi x}{L} \right) \) begins positive (above $y=0$ line) and ends negative (below line).

The original function begins negative and ends positive.

[2] (70 points) A one-dimensional particle in a rigid box extending from 0 to L is described by the wave function $\psi(x) = A \sin(\pi x/L)$, with quantum no. $n = 5$.

(a) (15 pts) Sketch the form of this wave function, as well as the probability of finding the particle at a given $x$:
(b) (15 pts.) Write down an expression for the normalization constant \( A \) in terms of a definite integral, but you need not evaluate the integral.

\[
\int_0^L \psi_0^*(x) \psi_0(x) \, dx = 1
\]

\[
\Rightarrow \quad A^2 \int_0^L \sin^2 \left( \frac{5\pi x}{L} \right) \, dx = 1
\]

\[
\Rightarrow \quad A = \sqrt{\frac{2}{L}}
\]

(c) (15 pts.) Write down an expression for the average value of the momentum of the particle as obtained over many measurements, simplifying as far as you can without actually evaluating any integral. [Note: \( d(sinax)/dx = acosx \)]

\[
\langle p \rangle = \int_0^L \psi_0^*(x) \left( -i \frac{\hbar}{2m} \right) \psi_0(x) \, dx \quad \psi_0(x) = A \sin \left( \frac{5\pi x}{L} \right)
\]

\[
\langle p \rangle = -i \frac{\hbar}{2m} \int_0^L A^2 \sin^2 \left( \frac{5\pi x}{L} \right) \cos \left( \frac{5\pi x}{L} \right) \, dx
\]

\[
= -i \frac{\hbar}{2m} \int_0^L A^2 \frac{5\pi}{L} \int_0^L \sin \left( \frac{5\pi x}{L} \right) \cos \left( \frac{5\pi x}{L} \right) \, dx
\]

\[
= -i \frac{\hbar}{2m} \frac{5\pi}{L} \int_0^L \sin \left( \frac{5\pi x}{L} \right) \cos \left( \frac{5\pi x}{L} \right) \, dx = 0
\]

(d) (15 pts.) Now use the symmetry of the problem and the functions involved to derive, without doing any mathematics, the answer to part (c).

One way: Particle moves to left as often as it moves to right in the box, so average momentum will be zero.

Another way (using functions above)

\[
\sin y \cos y = \frac{1}{2} \sin(2y) \Rightarrow \int_0^L \sin \left( \frac{10\pi x}{L} \right) = 0
\]

(e) (10 pts.) Now use the Uncertainty Principle and an approximate argument to estimate the minimum uncertainty in momentum of this particle in terms of \( L \).

Simple way: \( \Delta p \Delta x \geq \frac{\hbar}{2} \). Assume position of \( x \) is "unknown" inside box, so \( \Delta x = L \) (uncertainty in position).

So estimate of \( \Delta p \approx \frac{\hbar}{2L} \).