1. Certain atoms such as Na$^{24}$ can spontaneously emit a gamma ray—an energetic “particle” of light. When an atom does this, it recoils in the opposite direction, and its mass decreases slightly. Suppose an atom of mass $m$ is initially at rest, and emits a gamma ray (moving at the speed of light, of course) along the $x$ axis. Let the atom’s recoil velocity be $u$. Find a formula for the energy $E$ of the gamma ray in terms of $u$ and $m$.

Atom before emission: mass $m$, velocity 0, momentum 0, energy $mc^2$
Gamma ray: energy $E$, momentum $p = E/c$
Atom after emission: mass $m'$, velocity $u$, momentum $-m'\gamma_u u$, energy $m'c^2\gamma_u$

Momentum conservation: $E/c - m'\gamma_u u = 0 \Rightarrow m' = E/c\gamma_u$
Energy conservation: $mc^2 = E + m'c^2\gamma_u = E + (E/c\gamma_u u)c^2\gamma_u = E + Ec/u$

Solve for $E$: $E = mc^2/(1 + c/u)$

2. A uranium nucleus is moving through the lab at a speed of $.5c$ in the positive $x$ direction. It splits (fissions) into a xenon nucleus and a strontium nucleus. In the uranium’s rest frame the xenon nucleus moves in the positive $x$ direction at a speed of $.4c$, and the strontium nucleus moves in the negative $x$ direction at a speed of $.2c$. What are the speeds of the xenon and strontium nuclei in the laboratory’s reference frame? [Note: the numbers I’ve given are not at all realistic; don’t worry about xenon or strontium moving that fast in real life.]

Uranium and xenon are moving in same direction relative to lab, so $u_{Xe}$ is positive, and $v$ is negative ($v$ is lab velocity in uranium’s frame). Hence

$$u'_{Xe} = \frac{u_{Xe} - v}{1 - \frac{u_{Xe}v}{c^2}} = \frac{.4c + .5c}{1 + (.4)(.5)} = \frac{3}{4}c$$

Uranium and strontium are moving in same direction relative to lab, so $u_{Sr}$ is negative. Hence

$$u'_{Sr} = \frac{u_{Sr} - v}{1 - \frac{u_{Sr}v}{c^2}} = \frac{-.2c + .5c}{1 - (.2)(.5)} = \frac{1}{3}c$$

Note: you can get the signs from physical reasoning. If the uranium nucleus is moving in the $+x$ direction and splits off a xenon nucleus moving in the $+x$ direction relative to the uranium, then the xenon has to be moving faster relative to the lab, so the speeds add. But if the uranium nucleus splits off a strontium nucleus moving in the opposite direction, the strontium has to be moving slower relative to the lab, so the speeds subtract.