Physics 9DA  Spring 2003  Midterm 1 Answers
(yellow)

Constants:  \[ c = 3 \times 10^8 \text{ m/s} \  \ g = 10 \text{ m/s}^2 \]

**Part I:** Give short (1-3 sentence) answers to four questions.

1. **What is the Principle of Equivalence?**
The laws of physics take the same form in any locally inertial (i.e., freely falling) frame of reference.

2. **How much energy would it take to accelerate an electron to the speed of light?**
An infinite amount: as \( v \) approaches \( c \), \( \gamma \) approaches infinity.

3. **A muon is created in the upper atmosphere, and heads towards Earth. The muon has a very short lifetime \( \tau \), and in nonrelativistic physics it would not have time to reach Earth before decaying. In the Earth’s rest frame, according to special relativity, the muon can reach Earth because its lifetime is lengthened by time dilation. In the muon’s frame, though, its lifetime is still just \( \tau \). From the point of view of that frame, how does the muon make it to Earth?**
Length contraction: in the muon’s rest frame, its lifetime is just \( \tau \), but the distance between the upper atmosphere and the Earth is contracted, so it doesn’t have as far to go.

4. **Why is it impossible to escape from inside the event horizon of a black hole?**
Relative to an observer just inside the horizon, the horizon is moving outward at the speed of light (although gravity has so distorted space that the horizon’s area still remains constant), so an object inside can never catch up.

5. **Which weighs more, a hot brick or an otherwise identical cold brick? Explain.**
A hot brick has greater internal energy (its atoms have extra kinetic energy). Since energy contributes to mass, the hot brick has greater mass, and since inertial and gravitational mass are equal, the hot brick weighs more.

6. **What is “relativity of simultaneity”?**
Two events at different locations that are simultaneous in one inertial frame of reference will not, in general, be simultaneous in another inertial frame moving at a constant velocity with respect to the first.
Part II: Do three problems.

1. A neutral pion (or $\pi^0$) is an unstable elementary particle, with a lifetime of $\tau = 8.4 \times 10^{-17} \text{ s}$ and a rest energy of $mc^2 = 135 \text{ MeV}$. A pion most commonly decays to two photons (“particles of light”).

Consider a neutral pion moving through the laboratory at a speed of $.8c$. For simplicity, call the direction it is moving the positive $x$ direction.

a. What is the moving pion’s lifetime as measured in the laboratory?

\[
\gamma = \frac{1}{\sqrt{1 - (0.8)^2}} = \frac{1}{0.6} = \frac{5}{3}
\]

\[
\tau' = \gamma \tau = 1.4 \times 10^{-16} \text{ s}
\]

b. At the end of its lifetime, the pion decays to produce two photons (and nothing else). Suppose these travel along the same axis as the pion, that is, one in the $+x$ direction and one in the $-x$ direction. In the pion’s rest frame, what are the energies of these two photons?

Let the two photons have momentum and energy $(p_1, E_1 = p_1c)$ and $(-p_2, E_2 = p_2c)$. In the pion’s frame, the initial momentum is zero, so the final momentum is $p_1 - p_2 = 0$. Hence $E_1 = E_2$. The initial energy is just $mc^2$, so by conservation of energy,

\[
E_1 + E_2 = 2E_1 = mc^2 = 135 \text{ MeV}
\]

\[
E_1 = E_2 = 67.5 \text{ MeV}
\]

c. Using the Lorentz transformation (or any other method you prefer), find the energies of the two photons from part b in the lab’s rest frame.

The Lorentz transform of energy is

\[
E' = \gamma (E - vp)
\]

and for a photon, $p = \pm E/c$. So for photon 1,

\[
E_1' = \gamma (E_1 - v p_1) = \frac{5}{3} (E_1 - (0.8c)(E_1/c)) = \frac{1}{3} E_1 = 22.5 \text{ MeV}
\]

For photon 2,

\[
E_2' = \gamma (E_2 - v p_2) = \frac{5}{3} (E_2 - (0.8c)(-E_2/c)) = 3E_2 = 202.5 \text{ MeV}
\]
2. Because of the Earth’s rotation, the equator moves at a speed of about .45 km/s relative to the poles.

a. Suppose a clock at the equator is initially synchronized with a clock at the North Pole. After a year, how far behind will the clock at the equator be? (An answer to two significant figures is good enough. You may want to use the fact that one year is approximately $3.2 \times 10^7$ s.)

$$v/c = (450 \text{ m/s})/(3 \times 10^8 \text{ m/s}) = 1.5 \times 10^{-6}$$

$v/c$ is very small, so

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2} \frac{v^2}{c^2} \approx 1 + 1.1 \times 10^{-12}$$

(Note: if you tried to use a calculator, it probably didn’t work—the numbers are too small. For very small numbers, approximations by hand are a good idea!)

So the time loss in a year is

$$(1.1 \times 10^{-12})(3.2 \times 10^7 \text{ s}) \approx 3.5 \times 10^{-5} \text{ s}$$

b. There is another, competing effect on clock rates. Because of centrifugal acceleration, the Earth bulges slightly at the equator, so a clock at the equator is slightly “higher,” and experiences a slightly smaller gravitational field. How much must the equator bulge—that is, how much higher must a clock be at the equator than the North Pole—for gravitational time dilation to cancel the special relativistic effect of part a? [This is, in fact, the amount of the real equatorial bulge, so clocks at sea level on the Earth’s surface remain synchronized.] You may want to use the fact that the acceleration due to gravity near the Earth’s surface is $g \approx 10 \text{ m/s}^2$.

Need the factor from $\gamma$ to cancel the factor from gravitational time dilation, that is,

$$1 + \frac{1}{2} \frac{v^2}{c^2} = 1 + \frac{gh}{c^2}$$

or $gh = \frac{1}{2} v^2$. Now, $\frac{1}{2} v^2 \approx .1 \text{ km}^2/\text{s}^2$ and $g = 10^{-2} \text{ km/s}^2$, so

$$h \approx (.1)/(10^{-2}) \text{ km} = 10 \text{ km}$$
3. At midnight on New Year’s Eve, fireworks are set off simultaneously (in the Earth’s reference frame) in Davis and San Francisco, at sites 150 km apart (in the Earth’s frame). The pilot of an experimental rocket flying between San Francisco to Davis observes that his instruments show a time difference of $5 \times 10^{-4}$ s between the two explosions.

a. How fast is the rocket flying?

Earth frame $S$: $\Delta t = t_{\text{Davis}} - t_{\text{SF}} = 0$

$\Delta x = x_{\text{Davis}} - x_{\text{SF}} = 150 \text{ km} = 1.5 \times 10^5 \text{ m}$

Rocket frame $S'$: $\Delta t' = t'_{\text{Davis}} - t'_{\text{SF}} = \pm 5 \times 10^{-4}$ s

The ± is because we don’t yet know which is first, $t_{\text{Davis}}$ or $t_{\text{SF}}$.

$$\Delta t' = \gamma (\Delta t - \frac{v}{c^2} \Delta x) = \frac{\gamma}{c} \left( \frac{1.5 \times 10^5 \text{ m}}{3 \times 10^8 \text{ m/s}} \right) = \frac{\gamma}{c} \cdot (5 \times 10^{-4} \text{ s})$$

But we’re given that $\Delta t' = \pm 5 \times 10^{-4} \text{ s}$, so we have to choose the − sign. Then

$$\gamma \frac{v}{c} = 1 = \frac{v/c}{\sqrt{1 - (v/c)^2}}$$

$$1 - \frac{v^2}{c^2} = \frac{v^2}{c^2}$$

$$v = \frac{1}{\sqrt{2}} c \approx .707 c$$

b. If the rocket is flying from San Francisco toward Davis, which of the explosions happens first in the rocket’s frame?

We saw from part a that the ± must be −: that is, there was only a solution if $\Delta t' = t'_{\text{Davis}} - t'_{\text{SF}} < 0$. So $t'_{\text{Davis}} < t'_{\text{SF}}$, and the explosions at Davis happen at a smaller time, that is, earlier. In the rocket’s frame, the fireworks happen first in Davis.
4. A new unmanned rocket is launched from Earth at a speed of \( \frac{2}{3}c \). The design is flawed, and the rocket explodes. In the rocket’s rest frame, the explosion sends out an expanding sphere of debris, moving at half the speed of light.

a. To an observer on Earth, directly underneath the rocket, how fast is the fastest piece of debris moving, and in which direction? What about the slowest piece?

Velocities add as

\[
u' = \frac{u - v}{1 - \frac{uv}{c^2}}
\]

Here \( v = -\frac{2}{3}c \) (velocity of Earth in rocket’s frame), and \( u \) varies from \( \frac{1}{2}c \) (velocity of fragment moving up in rocket’s frame) to \( -\frac{1}{2}c \) (for fragment moving down). So the range of velocities in the Earth observer’s frame is from

\[
\frac{1/2 - 2/3}{1 - (1/2)(2/3)} = -\frac{1}{4}c \text{ (moving down)}
\]

to

\[
\frac{1/2 + 2/3}{1 + (1/2)(2/3)} = \frac{7}{8}c \text{ (moving up)}
\]

b. To the observer on Earth, is the expanding shell of debris spherical? Explain. It’s not. Compare the velocity of the rocket, the top piece of debris, and the bottom piece—the rocket is not in the center any more, as seen from Earth.
5. “Pushing an atom with light”: An atom of mass $M$ is initially at rest. An experimenter shines a strong pulse of light along the $x$ axis onto the atom, and the atom completely absorbs the light. Afterwards, the atom is observed to have a mass $M'$ and a velocity $v$ along the $x$ axis. Let $E$ and $p$ be the energy and momentum of the pulse of light.

a. What is $M'$ in terms of $M$ and $E$?

b. What is $v$ in terms of $M$ and $E$?

Initial: Energy $E_{atom} + E_{light} = Mc^2\gamma + E$
Momentum $p_{atom} + p_{light} = E/c$

Final: Energy $M'c^2\gamma$
Momentum $M'v\gamma$

So by conservation, $M'v\gamma = E/c$ and $M'c^2\gamma = Mc^2 + E$, or

$$M'c^2\gamma = Mc^2 + E, \quad M'vc\gamma = E$$

First do part b:

$$\frac{M'vc\gamma}{M'c^2\gamma} = \frac{v}{c} = \frac{E}{Mc^2 + E}$$

that is,

$$v = \frac{Ec}{Mc^2 + E}$$

Next find $\gamma$: from the equation above,

$$1 - \frac{v^2}{c^2} = 1 - \frac{E^2}{(Mc^2 + E)^2} = \frac{(Mc^2 + E)^2 - E^2}{(Mc^2 + E)^2} = \frac{M^2c^4 + 2Mc^2E}{(Mc^2 + E)^2}$$

so

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \frac{Mc^2 + E}{\sqrt{M^2c^4 + 2Mc^2E}}$$

Then, since from the original conservation equation

$$M'c^2 = \frac{Mc^2 + E}{\gamma}$$

we can read off

$$M'c^2 = \sqrt{M^2c^4 + 2Mc^2E}$$

or

$$M' = \sqrt{M^2 + 2ME/c^2}$$