Physics 116C Problem Set 2
Additional Problems

1. A cosine function \( f_1(t) = A \cos(2\pi f_0 t) \) is multiplied by \( f_2(t) \), a unit boxcar function (height=1) extending from \(-T_1\) to \(T_1\) (assume \( T_1 \gg 1/f_0 \)) to give \( f_3(t) = f_1(t)f_2(t) \). (Refer to figures in Fourier transform handout for mathematical definition of boxcar function. Note that \( f_0 \) and \( T_1 \) are constants.)

(a) Write down the corresponding Fourier transforms, \( F_1(f) \) and \( F_2(f) \).

(b) Use the convolution theorem to find a mathematical expression for the Fourier transform, \( F_3(f) \).

(c) Sketch the real part of the Fourier transform of \( F_3(f) \).

Notes: recall that \( \int_{-\infty}^{\infty} f(y)\delta(x-y)dy = f(x) \). The convolution integral \( g(x) \) of two functions \( f_1(x) \) and \( f_2(x) \) is defined as \( g(x) = \int_{-\infty}^{\infty} f_1(y)f_2(x-y)dy \).

2. Human hearing typically extends up to 17 kHz or so. The dynamic range between the softest (ppp) and loudest (fff) passages of classical music is 60 dB.

(a) Explain the advantages of digitizing an audio waveform of a musical performance at a sampling rate of 44 kHz and 16 bit resolution rather than 22 kHz and 8 bit resolution. Be quantitative.

(b) Explain why a low pass filter is required before digitization.

   i. What is the maximum frequency allowed to pass this filter for the 22 kHz sampling rate?

   ii. Would a simple single stage RC filter with corner frequency equal to this maximum frequency be sufficient? Explain. (Assume there is significant audio power extending beyond 17 kHz.)

   iii. What would be the result on the reproduced waveform of power remaining at frequencies beyond the desired maximum?

   iv. For a 22 kHz sampling rate, what would be the apparent frequency of a digitized 30 kHz signal if there were no filter?
1. (a) \( f_1(t) = A \cos(2\pi ft + \phi) \)
\[ F_1(f) = \frac{A}{2} \left[ \delta(f - f_0) + \delta(f + f_0) \right] \]
\[ f_2(t) = \begin{cases} 
1 & |t| < T_1 \\
\frac{1}{2} & |t| = T_1 \\
0 & |t| > T_1 
\end{cases} \]
\[ F_2(f) = 2T_1 \frac{\sin(2\pi T_1 f)}{2\pi T_1 f} \]

(b) \[ \text{Diagram of } f_1(t) \text{ and } f_2(t) \]

(c) The F.T. of a product \( f_1(t)f_2(t) \)
is the convolution \( F_1(f) \ast F_2(f) \)
\[ f_1f_2 = f_2f_1 \]
\[ g(f) = \int 2T_1 \frac{\sin(2\pi T_1 f')}{2\pi T_1 f'} \left[ \frac{A}{2} \delta(f - f' - f_0) + \frac{A}{2} \delta(f - f' + f_0) \right] df' \]
\[ = AT_1 \frac{\sin(2\pi T_1 (f - f_0))}{2\pi T_1 (f - f_0)} + AT_1 \frac{\sin(2\pi T_1 (f + f_0))}{2\pi T_1 (f + f_0)} \]
2. (a) 44 kHz and 16 bits give greater dynamic range and wider bandwidth than 22 kHz and 8 bits.

8 bits: 44 kHz $\Rightarrow f_c = 22$ kHz, covering the full audio spectrum.

22 kHz $\Rightarrow f_c = 11$ kHz, within the audio range, so audible high frequencies would have to be filtered out.

Dynamic Range: n bits

Range of largest to smallest signal is

$$\frac{2^n - 1}{2^n} < 2^n$$

for peak-to-peak signals. The ratio of amplitudes is the same.

In db, this is

$$R(db) = 20 \log_{10} 2^n = 20 n \log_2 2 = 6n$$

- 48 dB for 8 bits
- 96 dB for 16 bits

For orchestral music, Fff ~ 100 dB at 1 kHz  
PPP ~ 40 dB

so 8 bits is insufficient for high quality playback whereas 16 bits is reasonable.

(b) i) $f_c = \frac{f_{\text{sample}}}{2} = 11$ kHz

ii) No. The amplitude is only down by $\frac{1}{\sqrt{2}}$ at the RC filter corner frequency, leaving the possibility of significant power above $f_c$.

iii) Frequencies $> f_c$ would be aliased to $f < f_c$ causing distortion of the reproduced signal.

iv) Refer to notes: $f_{\text{observed}} = f - Nf_c$ (even n)  
$= 20$ kHz - 22 kHz = 8 kHz