Sampled signals, sampling theorem and aliasing:

\[ h(t) \Rightarrow h_n \quad \{ \begin{align*}
& n = 0, 1, \ldots, N-1 \quad (N \text{ samples}) \\
& \text{at time } t = nT \\
& \text{(measure time in units of } T) 
\end{align*} \]

Define Nyquist critical frequency \( f_c = \frac{1}{2T} \) (half the sampling frequency)

Note that sine wave at \( f_c \) has two samples/cycle (minimum necessary to recognize the component; and if phase were just right (or just wrong, rather) could not see it at all).

**Sampling Theorem**

If \( h(t) \) (continuous function) is sampled at intervals \( T \) and is band-width limited to frequencies less than or equal to \( f_c \), i.e., \( H(f) = 0 \) for all \( |f| > f_c \), then \( h(t) \) can be reconstructed uniquely from the samples. (But note in principle that the number of samples may be very large). In particular,

\[
h(t) = T \sum_{n=\infty}^{\infty} h_n \frac{\sin [2\pi f_c (t-nT)]}{\pi (t-nT)}
\]

[Shorthand: \( \text{sinc } x = \sin x \)]

*Numerical Recipes (see Ref. 5)*
Aliasing:

If $h(t)$ is not bandwidth limited appropriately, all the frequency components outside $-f_c < f < f_c$ are mapped into this range (aliasing):

$T \rightarrow$ freq. = $f' > f_c$

*Observed* $f'$:

\[
\begin{align*}
\text{if } f' &= \frac{3}{4T} \\
\text{Samples look like } f_{obs} &= \frac{1}{4T}
\end{align*}
\]

In general,

\[
\begin{cases}
\text{if } f_{obs} = f' - nf_c \quad \text{(even } n) \\
= -f' + (n+1)f_c \quad \text{(odd } n)
\end{cases}
\]

Some unique $n$ will bring $f_{obs}$ into $0 \leq f_{obs} < f_c$

Check:

Here, $n=1$ and $f_{obs} = -f' + 2f_c = -\frac{3}{4T} + \frac{1}{T} = \frac{1}{4T}$ (04)

In general, may need to filter frequencies - e.g., for audio response to 17 kHz with sample frequency of 44 kHz, must apply a sharp cutoff filter to eliminate $f > f_c = 22$ kHz ... although little power remains in this part of the spectrum.

References:

Press, et al., *Numerical Recipes*, Ch. 12 (M.R.)

Brigham, *The Fast Fourier Transform and its Applications*