Sampled Signals and Fourier Transform

\[ h_n = h(t_n) \]

\[ h(t) = h_n \quad \text{for} \quad n = 0, 1, \ldots, N-1 \] (N samples)

\[ \text{at time } t = nT \]

(measure time in units of \( T \))

Define Nyquist critical frequency \( f_c = \frac{1}{2T} \)

(half the sampling frequency)

Note that sine wave at \( f_c \) has two samples/cycle

(minimum necessary to recognize the component — and if phase were just right (or just wrong, rather) could not see it at all).

**Sampling Theorem**

If \( h(t) \) (continuous function) is sampled at

intervals \( T \) and is band-width limited to frequencies less than or equal to \( f_c \), i.e., \( \text{HT}(f) = 0 \) for all \( |f| > f_c \) then \( h(t) \) can be reconstructed uniquely from the samples.

(But note in principle that the number of samples may be very large). In particular,

\[ h(t) = T \sum_{n=-\infty}^{\infty} h_n \frac{\sin [2\pi f_c (t-nT)]}{\pi (t-nT)} \]

(Shorthand: \( \text{sinc} x = \frac{\sin x}{x} \))

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Numerical Recipes (see refs.)
Aliasing

If $h(t)$ is not band-width limited appropriately, all the frequency components outside $-f_c < f < f_c$ are mapped into this range (aliasing):

$$kT + freq. = f' > f_c$$

etched as observed

$$f_c = \frac{-3}{4T}$$

samples look like

$$f_{obs} = \frac{1}{4T}$$

In general, \[ f_{observed} = f' - nf_c \quad \text{(even n)} \]

\[ = f' + (n+1)f_c \quad \text{(odd n)} \]

Some unique \( n \) will bring \( f_{obs} \) into \( c \cdot f_c \)

Check: Here, \( n = 1 \) and \( f_{obs} = -f' + 2f_c = -\frac{3}{4T} + \frac{1}{T} = \frac{1}{4T} \)

In general, may need to filter frequencies - e.g., for audio response to 17 kHz with sample frequency & 44 kHz, must apply a sharp cutoff filter to eliminate \( f > f_c = 22 \) kHz ... although little power remains in this part of the spectrum.
Fourier transform (in a form convenient for sampled data)

\[ H(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i ft} \, dt \]
\[ h(t) = \int_{-\infty}^{\infty} H(f) e^{2\pi i ft} \, df \]

or equivalently,

\[ H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} \, dt \]
\[ h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} \, d\omega \]

Note Numerical Recipes has \(i\rightarrow-i\). Some texts differ in placements of a factor \(\frac{1}{2\pi}\).

Further theorem! (similar to coefficients for complex Fourier series)

If \(h(t)\) is real \(\Rightarrow H(-f) = H(f)^*\)

and every \(H(f)\) is real \& even odd, \(H(f)\) is imaginary odd.

Convolution

Define convolution \(g \ast h\) for \(g(t)\), \(h(t)\):

\[ g \ast h = \int_{-\infty}^{\infty} g(\tau) h(t-\tau) \, d\tau \quad \text{(fun of } t) \]

Convolution theorem:

\[ G(f) = \text{F.T. of } g(t), \quad H(f) = \text{F.T. of } h(t), \]

then \(g \ast h = \text{Fourier transform of } G(f) H(f)\)

(Product in freq. domain)

Convolution in time domain

and similarly for

\[ g(t) h(t) \underset{\text{F.t.}}{\longleftrightarrow} G(f) H(f) \]
Parseval's Theorem:
Total power in spectrum remains the same if calculate in time or frequency domain.

\[ \text{Total Power} = \int_{-\infty}^{\infty} |h(t)|^2 \, dt = \int_{-\infty}^{\infty} |H(f)|^2 \, df \]

One-sided power spectral density:
\[ P_h(f) = |H(f)|^2 - |H(-f)|^2 \quad 0 \leq f < \infty \]
\[ = 2|H(f)|^2 \quad \text{if } h(t) \text{ is real.} \]
\[ \text{(Total power} = \int_{0}^{\infty} P_h(f) \, df) \]

Some typical transform pairs - see figures in text by Brigham (using their F.T. conventions).

In particular, for "boxcar" function
\[ h(t) = A \quad 0 \leq t \leq 2\pi \]
\[ = 0 \quad \text{otherwise} \]
\[ |H(f)| = 2A/T_0 \left| \frac{\sin(2\pi T_0 f)}{2\pi T_0 f} \right| \]
(See Brigham, p.14)

and for series of \( \delta \)-facs equally spaced in time,
\[ h(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \iff H(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T}) \]
get corresponding series of \( \delta \)-facs equally spaced in \( f \).
Figure 2.12 Catalog of Fourier transform pairs.
Sampled waveforms can be considered as products of series of \( \delta \)-func spaced at \( T = 1/f_c \) with the waveform \( h(t) \). Assume \( H(f) \) is bandwidth limited so \( H(f) = 0 \) for \( f > f_c = 1/2T \). The resulting F.T. of the sampled waveform is the convolution of \( H(f) \) with the F.T. of the \( \delta \)-func, more \( \delta \)-func spaced by \( T \). In particular then, the original \( H(t) \) will extend between \( -f_c \) and \( f_c \) about \( f_c = 0 \). But it will be replicated about every \( \delta \)-func from the F.T. of the sampling \( \delta \)-func series. Thus, we multiply the complete \( H(f) \) of the sampled waveform by the boxcar func extending from \( -f_c \) to \( f_c \) in the frequency domain, to select only the desired frequencies. The resulting \( h(t) \) is then the convolution of this boxcar function F.T. in the time domain with the original sampled waveform.

The result is therefore

\[
 h(t) = \sum_{n=-\infty}^{\infty} h(nT) \delta(t - nT) \sum_{n=-\infty}^{\infty} \frac{\sin(2\pi f_c (t - T))}{2\pi f_c (t - T)} \cdot \frac{\sin(2\pi f_c (t - nT))}{\pi (t - nT)} 
\]

(proof of sampling theorem).
If the $H(f)$ is not bandwidth limited, the boxcar will pick up tails of neighboring distributions and lose its own higher frequency tails - they will effectively become aliased to false Fourier components within the ±$f_T$ interval.

### Discrete Fourier Transform

$h_k = h(t_k), \quad t_k = kT \quad k = 0, \ldots, N-1$

Estimate frequency spectrum at

$f_n = \frac{n}{NT} \quad n = -\frac{N}{2}, \ldots, \frac{N}{2}$

$$H(f_n) = \int_{-\infty}^{\infty} h(t) e^{-2\pi j f_n t} dt = \sum_{k=0}^{N-1} h_k e^{-2\pi j f_n t_k}$$

$$= \frac{1}{T} \sum_{k=0}^{N-1} h_k e^{-2\pi j k n/N} = TH_n \quad (H_n \text{ is defined as the discrete Fourier transform})$$

$$H_n = \sum_{k=0}^{N-1} h_k e^{-2\pi j k n/N} \quad \text{And similarly for inverse transform}$$

(see Essick, Ch. 9)

Fast Fourier transform uses numerical algorithm (usually called Cooley-Tukey but was noted earlier by Danielson & Lanczos, for example) ... reduces from a calculation which grows like $N^2$ where $n$ is the number of samples to $N \log_2 N$ ... great help in calculation.

FFT can be used to estimate power spectra (but note effects of finite-length samples ... sampling them assumed $\frac{2\pi}{T}$ - see Near R.)

* See next page

(Also, see Essick, Ch. 9 on leakage and windowing)
And filter functions can be applied in the \textit{f} domain to do filtering on the sampled signal in the \textit{t} domain (can't eliminate aliasing effects, however). Of course filter will have an effect on the time response much as we saw with analog filters.

Also, real-time filtering techniques can be developed with sampled spectra. (See Brigham.) (If real-time is not an issue, FFT techniques are probably more straightforward.)

Other related topics: deconvolution, correlation functions.

* Using only part of an infinite series amounts to multiplying by a boxcar function again resulting in a convolution of the original frequency spectrum with a sinc function in the frequency domain - sharp spikes get smeared out.


Brigham, \textit{The Fast Fourier Transform and its Applications}