Problem 1 solution:

The data were fitted with the function \( y = a_0 + a_1 x \). The results of the LabVIEW Levenberg-Marquardt fit are shown below:

(a) Write the results for \( a_0 \) and \( a_1 \), each in the form \( xxx \pm yyyy \), where \( xxx \) represents the value (to a reasonable number of significant figures) and \( yyyy \) represents the standard deviation.

\[ a_0 = 66.0 \pm 2.9 \quad a_1 = 66.0 \pm 2.6 \]

(b) The point at \( x_i = 300 \) appears to miss the fitted line by 2.29\( a_1 \). What is the probability of a deviation this large or larger (in either direction)? Hint: remember Table C.2 in Bevington.

\[ p(x_i - 2.29a_1 < x_i < x_i + 2.29a_1) = 0.954 \quad \Rightarrow \quad p = 0.026 \quad \Rightarrow \quad P = 2.2 \times 10^{-2} \quad (2.2\%)

(c) How many degrees of freedom (\( \nu \)) are there for this fit?

7 points - 3 parameters \( \Rightarrow \nu = 7 - 2 = 5 \)

(d) Find \( \chi^2/\nu \) for the fit. What is the expectation value for this quantity?

\[ \chi^2/\nu = 1.28 \Rightarrow \chi^2 = 5 \times 1.28 = 1.92 \quad \Rightarrow \quad \langle \chi^2/\nu \rangle = 1 \]

(e) Use Table C.4 in Bevington to estimate the probability of exceeding this value of \( \chi^2/\nu \).

\[ 1,45 \% \quad \Rightarrow \quad p = 0.20 \quad \Rightarrow \quad P \approx \frac{0.20}{1.45} \approx 0.13\]

(f) How you can tell that the errors in \( a_0 \) and \( a_1 \) are correlated? What is the value of \( \sigma_{a0}^2 \)?

Covariance elements of covariance matrix not negligible.

\[ \sigma_{a0}^2 = 0.2 \quad \sigma_{a1}^2 = 1.5 \times 10^{-2} \]

(g) We calculate the area under the curve from \( x = 0 \) to \( x = 600 \) using the formula

\[ A = a_0 \Delta x + \frac{1}{2} a_1 (\Delta x)^2 \]

where \( \Delta x = 600 \). Find an expression for \( \sigma_A^2 \), the variance of \( A \). You do not need to evaluate this expression but give numerical values for all quantities entering into the calculation. Covariance not negligible.

This is of the form \( x = ax + bv \) where \( a = \Delta x, b = \frac{1}{2} \Delta x^2 \)

\[ \sigma_A^2 = \sigma_{a0}^2 + \sigma_{a1}^2 + \frac{1}{2} (\Delta x)^2 \sigma_{a1}^2 \]

\[ \Delta x = 600, \sigma_{a0}^2 = 6.0, \sigma_{a1}^2 = 5.03 \times 10^{-2}, \sigma_{a1}^2 = -1.5 \times 10^{-2} \]

\[ \sigma_A^2 = 2.39 \times 10^{-6} + 1.62 \times 10^6 - 1.67 \times 10^6 = 10^{-6} \quad \Rightarrow \quad A = 61.7 \pm 6.2 \times 10^{-4} \]

Conclusion:

\[ A = (61.7 \pm 1.5) \times 10^3 \]
Problem 2 Solution:

Transformation method: let \( r_i \) be a sample from a uniform distribution in the range \([0,1]\). Then a sample \( x_i \) from the \( \chi^2 \) distribution with \( \nu \) degrees of freedom is given by

\[
x_i = F^{-1}(r_i; \nu)
\]

where \( F^{-1} \) is the inverse cumulative distribution function of the \( \chi^2 \) distribution with \( \nu \) degrees of freedom. (The range of \( F(x; \nu) \) is from 0 to 1; \( x \) is the value for which \( F(x; \nu) = r \).

The \( \chi^2 \) distribution with \( \nu \) degrees of freedom has mean = \( \nu \) and variance = \( 2\nu \). It approaches a Gaussian for \( \nu \gg 1 \). Here is a plot for \( \nu = 40 \). The sample mean is 39.94, quite close to 40, and the standard deviation is 8.942, very close to the square root of 80. The distribution is beginning to look Gaussian but is still obviously not symmetrical (has non-zero skewness). Also, the mean is only 4.5 standard deviations away from 0 and \( \chi^2 \) cannot be negative.
Problem 3 Solution:

(a) Find an expression for $H_2(f)$ in terms of an integral and evaluate the integral if possible.

$$H_2 = F(G) = \frac{A}{2} \left[ \delta (f - f_c) + \delta (f + f_c) \right]$$

$$\alpha = \frac{1}{2} \left[ \delta (f - f_m) + \delta (f + f_m) \right]$$

$$H_2 = F \star G = \int_{-\infty}^{\infty} \frac{df'}{2\pi} \int_{-\infty}^{\infty} \frac{df''}{2\pi} \left[ \delta (f' - f_c) + \delta (f' + f_c) \right] \times \left[ \delta (f'' - f_m) + \delta (f'' + f_m) \right]$$

$$= \frac{A}{4} \int_{-\infty}^{\infty} \frac{df'}{2\pi} \delta (f' - f_c) \left[ \delta (f'' - f_m) + \delta (f'' + f_m) \right] +$$

$$+ \frac{A}{4} \int_{-\infty}^{\infty} \frac{df'}{2\pi} \delta (f' + f_c) \left[ \delta (f'' - f_m) + \delta (f'' + f_m) \right]$$

$$= \frac{A}{4} \left[ \delta (f - f_c - f_m) + \delta (f - f_c + f_m) \right] +$$

$$+ \frac{A}{4} \left[ \delta (f + f_c - f_m) + \delta (f + f_c + f_m) \right]$$

(b) Let $f_c = 1$ MHz and $f_m = 1$ kHz. At what positive frequencies are $\delta$ functions located due to $H_2(f)$?

$$H_2(f) = \frac{A}{4} \left[ \delta (f - 1000\text{ kHz} - 1\text{ kHz}) + \delta (f - 1000\text{ kHz} + 1\text{ kHz}) +$$

$$+ \delta (f + 1000\text{ kHz} - 1\text{ kHz}) + \delta (f + 1000\text{ kHz} + 1\text{ kHz}) \right]$$

(c) Consider a modulating signal at 6 kHz. It would produce a delta function 6 kHz away from the original carrier frequency. If the next station’s carrier were 10 kHz away, one delta function would be within 4 kHz of that carrier, inside its 5 kHz range. In other words, it would interfere with the other station’s transmission.
Problem 4 Solution: *Note* the use of \( f(t) \) instead of \( h(t) \) in the solution. Be careful not to confuse this with the \( f \) variable representing frequency in \( F(f) \)!

Notation change: the solution uses \( f \) instead of \( h \) and \( F \) instead of \( H \).

\[
\text{Problem 4 Solution: Note the use of } f(t) \text{ instead of } h(t) \text{ in the solution. Be careful not to confuse this with the } f \text{ variable representing frequency in } F(f)!
\]

\[
(f(t) = A \cos(2\pi f_0 t) \\
F_1(f) = \frac{A}{2} \delta(f_f) + \frac{A}{2} \delta(f_f) \\
\delta(t) = \begin{cases} \\
1 & \text{if } |t| < T_1 \\
0 & \text{if } |t| > T_1 \\
\end{cases} \\
F_2(f) = 2T_1 \frac{\sin(2\pi T_1 f)}{2\pi T_1 f} \\
\text{(c) The } \delta \text{ product } f(t) \delta(t) = f(t) \\
\text{is the convolution } F_1(f) \ast F_2(f) = F_3 \\
F_3(f) = \int 2T_1 \frac{\sin(2\pi T_1 f)}{2\pi T_1 f} \left[ \frac{A}{2} \delta(f - f_f) + \frac{A}{2} \delta(f + f_f) \right] df \\
= A T_1 \frac{\sin(2\pi T_1 (f_f - f_f))}{2\pi T_1 (f_f - f_f)} + A T_1 \frac{\sin(2\pi T_1 (f_f + f_f))}{2\pi T_1 (f_f + f_f)} \\
\text{(d)}
\]

(e) Taking a finite sequence of samples is equivalent to multiplying the entire sequence by a boxcar function as in part (a). Thus the Fourier transform of the cosine will be convolved with the Fourier transform of the boxcar, sinc(f). Examples involving finite Fourier transforms (FFT's) are shown on p. 267 of Bevington. As shown in the text, the appearance of the FFT will depend on details of the relationship between the number of samples, the sampling frequency and the frequency of the waveform. In general, the peak gets spread out over several bins ("leakage").