Physics 116A Notes
Fall 2003
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• References:
  – Text for course:
  – Others as noted
Physics 116A, 10/13/03: Outline

- Assignment 3
- Differential equation relation of voltage and current for capacitor (done already) and inductor
- Transient response for RC circuit
- Steady-state (forced) response to sinusoidal input voltage and Alternating Current (AC) circuit analysis
  - Complex $i$ and $v$ to find ODE steady-state solutions
  - Complex impedance, $Z$
  - Extension of linear circuit analysis to AC circuits
  - **Example**: Low pass RC filter and Bode plot for lab
- Next: *Phasor* representation of (real or complex) sinusoids
Assignment 3, due 10/20/03

- Read 5.2-5.4:
  - resonance
  - complex frequency
  - linear systems with feedback
    (We’ll return to feedback near the end of 116A)

- We also touch on 5.5-5.7:
  - Significance of complex frequency, s
  - Intro. to Laplace transforms applied to circuit analysis
    * Detailed applications of Laplace transforms left for 116B

Problem assignment— Ch. 3: 3.2, 3.9; Ch. 4: 4.6, 4.8, 4.10, 4.12(a), 4.17, 4.28; Ch. 5: 5.1.

- Notes:
  - In Prob. 3.2(a), find $v_L(t)$. Note that $i(t)$ is given.
Capacitor Basics

\[ Q = CV_c \]

\[ i = C \frac{dV_c}{dt} \]

Since the magnitude of \( Q \) is the same on the two plates, the current flowing in from node a equals the current flowing out through node b.

Note no net charge is collecting on capacitor

*Stray capacitance* is neglected.
**Capacitors (continued)**

\[ i = C \frac{dv_C}{dt} \]

- Capacitor is open circuit for DC (blocks DC)

- Integral relation for capacitor voltage:
  \[ v_C(t) = \frac{1}{C} \int_{-\infty}^{t} i(t') \, dt' \]

- The voltage across a capacitor can’t change instantaneously
  - unless we have a delta function current spike

- Energy stored in capacitor
  \[ w_L(t) = \frac{1}{2} C v_C^2 \]
  - *Do you see how and where energy is stored?*
Inductor Basics

Inductor:

In the external circuit, we see a voltage across ab of

\[ V_L = L \frac{di}{dt} \]

What happened to Lenz’s law? It’s still there because \( V_L \) is a voltage drop. *  
Recall Faraday’s law of induction:

\[ \mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \]

Lenz’s law

Note \( V_L = V_{ab} = -\mathcal{E} = -\frac{d\Phi_B}{dt} = L \frac{di}{dt} \)

* as seen by the external circuit

self-inductance

(\( \Phi_B \propto i \))
Inductors (continued)

\[ v_L = L \frac{di}{dt} \]

• Inductor is a short circuit for DC

• Opening switch in series with inductor carrying current will cause spark
  – Do you see why?

• Integral relation for current in inductor:
  \[ i(t) = \frac{1}{L} \int_{-\infty}^{t} v_L(t') \, dt' \]

• The current through an inductor can’t change instantaneously
  – unless we have a delta function voltage spike

• Energy stored in inductor
  \[ w_L(t) = \frac{1}{2} Li^2 \]
  – Do you see how and where energy is stored?
RC Natural Response – once over lightly

\[ i = -C \frac{\mathrm{d}V_c}{\mathrm{d}t} \left\{ \text{note sign of } V_c \text{ since } i \text{ goes from } - \text{ to } + \right\} \]

\[ \frac{\mathrm{d}V_c}{\mathrm{d}t} = -\frac{1}{RC} V \quad \text{separable} \]

\[ \frac{1}{V} \frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{RC} \frac{\mathrm{d}t}{\mathrm{d}t} \]

\[ \ln V = -\frac{1}{RC} t + C \quad \text{of integration} \]

\[ V = Ae^{-t/RC} \quad \text{where } A = \frac{C}{R} \]

\[ \text{at } t = 0 \quad \Rightarrow A = V_0 \Rightarrow A = V_0 \]

\[ V = V_0 e^{-t/\tau} \quad \text{where } \tau = RC \]

\( \tau \) is called the time constant
RC Natural Response Plot

v(t) vs. t for RC discharge
\( \tau = RC = 1.0 \text{ ms}, v_0 = 10 \text{ V} \)
\( v = 37\% \text{ of } v_0 \text{ when } t = \tau \)

\( \tau \) and \( v(\tau) \) are indicated by the dashed lines.

Note that the graph has a title telling what it is, the axes are labeled and units are given...a word to the wise for the graphs in your lab logbook
AC Analysis: Sinusoidal Voltage Source

Circuits with $L, R, C$ driven by sinusoids

Find steady-state response (particular solution)

Use solutions of form $A e^{j \omega t}$ as before

(Note complex coefficient. $j = \sqrt{-1}$ (a.k.a. $i$))

Example:

\[ V_s = V_m \cos(\omega t) \]

$\omega = \text{angular frequency of source, a constant,}$

$V_m = \text{amplitude of sinusoid.}$
AC Analysis: Complex I, V

Introduce complex quantities to represent the voltages and currents to solve the D.F.
(reduce it to an algebraic equation)

\[ V_s = V e^{j\omega t} = V_m e^{j\theta_s} = V_m e^{j\phi_s} \]

where \( V_s = V_m e^{j\phi_s} \)

Connection with reality (in more ways than one):

\[ V_s = \Re_{\Re} V_s \quad \text{(real part is actual } V_s) \]

Since \( V_s = V_m \cos \omega t, \ \phi_s = 0 \text{ in this case.} \]
As a function of $t$, $\mathbf{v}_s$ traces a circle of radius $v_m$ in the complex plane at constant angular velocity $\omega$ while $\text{Re}(\mathbf{v}_s) = v_m \cos(\omega t + \phi)$ undergoes simple harmonic motion along the real axis.
AC: IV Relations for R, L, C

\[ \text{General: O.D. } i, v \Rightarrow \text{steady-state sinusoidal response w/ complex } V, I \]

\[ R: \quad \frac{1}{R} \quad \downarrow i(t) \quad \Rightarrow \quad u_R(t) = i_R \]

\[ C: \quad \downarrow \quad \frac{1}{C} \quad \Rightarrow \quad u_C(t) = \frac{1}{C} \frac{d}{dt} v(t) \]

\[ L: \quad \downarrow \quad \frac{1}{L} \quad \Rightarrow \quad u_L(t) = L \frac{d}{dt} i(t) \]

\[ \begin{cases} \text{for } V(t) = I(t) e^{j\omega t} & \Rightarrow \quad u_R(t) = I(t) e^{j\omega t} \\ u_L(t) = L I(t) e^{j\omega t} \end{cases} \]

\[ \begin{cases} \text{or } V(t) = I(t) R & \Rightarrow \quad u_R(t) = I(t) e^{j\omega t} \\ u_C(t) = \frac{1}{C} I(t) \end{cases} \]
AC: Complex Impedance

These relations are all of the form

\[ \mathbf{V} = \mathbf{I} \cdot \mathbf{Z} \]

\[ \mathbf{Z} \text{ is called the complex impedance} \]

where

\[ \mathbf{Z} = \mathbf{Z}_L = j\omega L \text{ for inductor} \]
\[ \mathbf{Z} = \mathbf{Z}_C = \frac{1}{j\omega C} \text{ for capacitor} \]
\[ \mathbf{Z} = \mathbf{R} \text{ for resistor} \]

and with complex \( \mathbf{I}, \mathbf{V}, \mathbf{Z} \), we get linear algebraic equations from Kirchhoff's laws for the steady-state sinusoidal solutions.

This method is called AC analysis with complex impedance (AC means Alternating Current)
Example: Low Pass RC Filter

Example:

Find \( V_0 \) \& \( R \)
\[ V_5 = V_m \cos \omega t \]

\( (V_m, \omega \text{ are const}) \)
\[ V_3 = V_2 + j V_0 \]
\[ V_0 = V_5 - j \omega RC V_0 \]
\[ V_0 = \frac{V_5}{1 + j \omega RC} \]

Define
\[ H(j \omega) = \frac{V_0}{V_5} \]
\[ \omega_c = \frac{1}{RC} \]

\( H(j \omega) \) called a forward voltage transfer function.
\( \omega_c \) is called the "corner" angular frequency.
\( f_c = \frac{\omega_c}{2\pi} \) called "corner frequency."

How would you modify this to make a high pass RC filter?
Now look at transfer function as we vary the frequency

- Find response (magnitude and phase)
  - far below the corner frequency
  - at the corner frequency
  - far above the corner frequency
• A sinusoid with \( \omega \ll \omega_c \) will have \( H(j\omega) \approx 1 \).

• If \( \omega = \omega_c \), \( H(j\omega_c) = \frac{1}{1+j} \)

  Polar representation of \( \frac{z}{1+j} \) (\( \Re(z) \) just stands for a complex number here, not impedance)

  \[
  z = r \cos \theta + j \sin \theta \\
  \theta = \tan^{-1} \frac{\Im(z)}{\Re(z)} = \tan^{-1} 1 = 45^\circ \ (\pi/4)
  \]

  \[
  z = r = 1 \Rightarrow e^{j \theta} = 1 \ e^{j \pi/4} = \sqrt{2} \ e^{j 45^\circ} \ (\text{another polar notation})
  \]

  \[
  H(j\omega_c) = \frac{1}{z} = \frac{1}{\sqrt{2}} \ e^{-j \pi/4} = \frac{\sqrt{2}}{2} (1-j) 4 - 45^\circ
  \]

  (again, \( z \) here just represents a complex number, as in math, not an impedance)

  At \( \omega_c \), the output sinusoid magnitude is 70.7% of the input magnitude (i.e., amplitude) and the phase has been shifted \(-45^\circ\).
Output as $\omega \to \infty$

$$|H(j\omega)| = \sqrt{H^*H} = \sqrt{\frac{1}{(1-j\omega\omega_c)(1+j\omega\omega_c)}} = (1+(\frac{\omega}{\omega_c})^2)^{-\frac{1}{2}}$$

$$\lim_{\omega \to \infty} |H(j\omega)| = 0$$

Also for $\omega \gg \omega_c$ $|H(j\omega)| \approx \frac{\omega_c}{\omega}$ \quad (since $\frac{(\omega}{\omega_c})^2 > 1$)

(falls like $\frac{1}{\omega}$)

The phase, $\phi$, of $H(j\omega)$ is $\phi = -\tan^{-1}\left(\frac{\omega}{\omega_c}\right)$ \quad (see next page)

so the phase shift of the network approaches $-90^\circ$ as $\omega \to \infty$

We summarize this in the **Bode plot**

(Hendrik W. Bode worked on stabilizing Harold S. Black's feedback amplifier at Bell Labs along with Harry Nyquist, whose plot we will encounter later.)
More on $\text{ang}(H)$

Note that the expression for $\phi$, the angle (or argument) of $H(j\omega)$, was given on the previous slide without justification. I'll provide that here (the text uses the notation “$\text{ang}$” in place of “$\text{arg}$”).

\[
H(j\omega) = \frac{1}{1 + j\omega/\omega_c}
\]

\[
\phi \equiv \text{arg}(H) = -\text{arg}(1 + j\omega/\omega_c)
\]

\[
\phi = -\tan^{-1}\left[\frac{\text{Im}(1 + j\omega/\omega_c)}{\text{Re}(1 + j\omega/\omega_c)}\right] = -\tan^{-1}\left(\frac{\omega}{\omega_c}\right).
\]

Sec. 4.2 in the text provides a nice review of complex number manipulations.
Bode Plot

Bode Plot: actually two plots, one above the other

Plot 1: \( A_v(\text{dB}) \) vs. \( \log_{10} \omega \) (or \( \log_{10} f \))

Plot 2: Angle of \( H(j\omega) \) vs. \( \log_{10} \omega \) (or \( \log_{10} f \)).

\[
A_v = |H(j\omega)| \quad \text{(called "voltage gain")}
\]

\[
= \left| \frac{\sqrt{25}}{\sqrt{25}} \right|
\]

\[
A_v(\text{dB}) = 20 \log_{10} \left| \frac{\sqrt{25}}{\sqrt{25}} \right|
\]

(this represents a power ratio into a constant load resistance since \( P \propto V^2 \))

Why use log-log plot for \( A_v(\omega) \)?
Bode Plot (continued)

Bel from A.G. Bell. Loudness of sound depends on power and the brain perceives a logarithmic scale. deciBel = \( \frac{1}{10} \) of a Bel.

Example for \( \omega = \omega_c \): \( A_v = \frac{V_2}{V_1} = 2 \log_{10} \frac{V_2}{V_1} \)

\[ = -10 \log_{10} 2 = -3 \text{dB} \]

(\( \omega_c \) is the "-3dB point")

For \( \omega >> \omega_c \): \( A_v \propto \frac{1}{\omega} \) "20 dB/decade"

"6 dB/octave" Look for these features on plots

Note that the corner frequency is the half-power point
Low Pass RC Filter Bode Plot

\( R = 1.0 \text{ k}\Omega, \ C = 0.010 \ \mu\text{F} \)

Why is \( f_c = \frac{1}{2\pi RC} = 16 \text{ kHz} \) called the “corner frequency?”
Loudspeaker Crossover Network
Examples of low pass, bandpass and high pass filters in action
to channel the proper frequencies into the appropriate drivers.
In the above, "mfd" means $\mu$F.