Part I: Short Answers (7 points each; do 4)
This section consists of five problems. You may do any four. (If you do all five, I will count whichever four you do best on.)

1. A particle in an infinite square well has a wave function
   \[ \psi(x) = \frac{3}{5} \psi_3(x) + c \psi_{11}(x) \]
   where \( \psi_n(x) \) are the (normalized) energy eigenstates. What is the magnitude \( |c| \) of the constant \( c \)?
   From normalization,
   \[ \left( \frac{3}{5} \right)^2 + |c|^2 = 1 \Rightarrow |c|^2 = 1 - \frac{9}{25} = \frac{16}{25} \]
   so \( |c| = \frac{4}{5} \).

2. Suppose that a particle is in an energy eigenstate, that is, \( \hat{H}\psi = E\psi \) for some constant \( E \). Find the uncertainty \( \Delta \hat{H} \) of the Hamiltonian, that is, the uncertainty in the energy. (Give a mathematical demonstration, not just hand-waving!)
   \[ (\Delta \hat{H})^2 = \langle \hat{H}^2 \rangle - \langle (\hat{H})^2 \rangle \]
   \[ \langle \hat{H} \rangle = \int \psi^* \hat{H} \psi dx = \int \psi^* E \psi dx = E \int \psi^* \psi dx = E \]
   \[ \langle \hat{H}^2 \rangle = \int \psi^* \hat{H}^2 \psi dx = \int \psi^* \hat{H} E \psi dx = E \int \psi^* \hat{H} \psi dx = E \int \psi^* E \psi dx = E^2 \int \psi^* \psi dx = E^2 \]
   So \( (\Delta \hat{H})^2 = E^2 - E^2 = 0 \), and thus \( \Delta \hat{H} = 0 \).

3. “Tunneling” described the passage of a quantum particle through a classically forbidden region (“barrier”). If a particle approaches a potential barrier from the left, the wave function to the far right of the barrier looks like
   \[ \psi(x) = t(x)e^{ikx} \quad (x \gg 0) \]
   where \( t(x) \) is a real (that is, not complex) function. Find the probability current for this wave function in this region \( (x \gg 0) \). Express your result in terms of the momentum of the particle.
   \[ J = \frac{\hbar}{2mi} \left( \psi \frac{\partial \psi}{\partial x} - \psi^* \frac{\partial \psi^*}{\partial x} \right) = \frac{\hbar}{2mi} \left( te^{-ikx} \frac{\partial}{\partial x} (te^{ikx}) - te^{ikx} \frac{\partial}{\partial x} (te^{-ikx}) \right) \]
   \[ = \frac{\hbar}{2mi} \left[ te^{-ikx} \left( \frac{\partial t}{\partial x} e^{ikx} + ikte^{ikx} \right) - te^{ikx} \left( \frac{\partial t}{\partial x} e^{-ikx} - ikte^{-ikx} \right) \right] \]
   \[ = \frac{\hbar}{2mi} [2ikt^2] = \frac{\hbar k}{m} t^2 = \frac{p}{m} t^2 \]
4. Suppose a harmonic oscillator is in the energy eigenstate $\psi_n(x)$. What is the expectation value of $\hat{x}\hat{p}\hat{x}$? Briefly explain how you got this answer.

$\langle \hat{x}\hat{p}\hat{x} \rangle = 0$. $\hat{x}$ and $\hat{p}$ are both monomials in the raising and lowering operators $\hat{a}_+$ and $\hat{a}_-$, so each term in $\hat{x}\hat{p}\hat{x}$ has three powers. Clearly, any such term must have unequal numbers of raising and lowering operators, so when applied to $\psi_n$, it will give a constant times some $\psi_m$ with $m \neq n$. By orthonormality, $\int \psi_n^*\psi_m dx = 0$ for $m \neq n$, so the expectation value is a sum of zeroes.

5. Suppose you are told that the initial wave function of an infinite square well at time $t = 0$ is a given function $\Psi(x,0)$. Describe the sequence of steps you could use to determine the time-dependent wave function $\Psi(x,t)$. (Don’t try to actually carry these steps out, but describe them clearly enough to show that you could carry them out, given time and integral tables.)

(a) Write

$$\Psi(x,0) = \sum_n c_n \psi_n(x)$$

where the $\psi_n$ are energy eigenstates.

(b) Extract the coefficients $c_n$:

$$c_n = \int \psi_n^* \Psi(x,0) dx = \sqrt{\frac{2}{a}} \int_0^a \sin \left( \frac{\pi nx}{a} \right) \Psi(x,0) dx$$

(c) Write

$$\Psi(x,t) = \sum_n c_n \psi_n(x)e^{-iE_n t/\hbar}$$

where $E_n = \frac{n^2 \hbar^2 \pi^2}{2ma^2}$ is the energy of state $\psi_n$. 

Part II: Problems (24 points each; do 3)
This section consists of four problems. You may do any three. (If you do more than three, I will count whichever three you do best on.)

1. Harmonic oscillator superpositions:
Suppose a harmonic oscillator has a wave function at \( t = 0 \) of
\[
\Psi(x, 0) = \frac{1}{\sqrt{2}} \psi_p(x) + \frac{1}{\sqrt{2}} e^{i\phi} \psi_{p+1}(x)
\]
where \( p \) is an integer, \( \phi \) is a constant phase, and \( \psi_p(x) \) is the \( p \)th energy eigenstate of the harmonic oscillator.

(a) Find \( \Psi(x, t) \) for an arbitrary time \( t \).
\[
\Psi(x, t) = \frac{1}{\sqrt{2}} \left( \psi_p e^{-iE_p t/h} + e^{i\phi} \psi_{p+1} e^{-iE_{p+1} t/h} \right)
\]
\[
= \frac{1}{\sqrt{2}} \left( \psi_p e^{-i\omega(p+\frac{1}{2})t} + \psi_{p+1} e^{i\phi - i\omega(p+\frac{1}{2})t} \right)
\]

(b) Find the expectation value \( \langle \hat{H} \rangle \) of the energy at time \( t \).
\[
\langle \hat{H} \rangle = \int \Psi^*(x, t) \hat{H} \Psi(x, t) dx
\]
\[
= \frac{1}{2} \int \left( \psi_p^* e^{iE_p t/h} + e^{-i\phi} \psi_{p+1}^* e^{iE_{p+1} t/h} \right) \left( E_p \psi_p e^{-iE_p t/h} + E_{p+1} e^{i\phi} \psi_{p+1} e^{-iE_{p+1} t/h} \right) dx
\]
\[
= \frac{1}{2} \int \left( \psi_p^* E_p \psi_p + \psi_{p+1}^* \psi_{p+1} \right) dx + \text{terms involving } \int \psi_p^* \psi_{p+1} dx \text{ or } \int \psi_{p+1}^* \psi_p dx
\]
\[
= \frac{1}{2} (E_p + E_{p+1}) = \hbar \omega (p+1)
\]
Alternatively,
\[
\langle \hat{H} \rangle = \sum |c_n|^2 E_n = \frac{1}{2} E_p + \frac{1}{2} E_{p+1}
\]

(c) Find the expectation value \( \langle \hat{x} \rangle \) of position at time \( t \).
\[
\langle \hat{x} \rangle = \int \Psi^*(x, t) \hat{x} \Psi(x, t) dx = \sqrt{\frac{\hbar}{2m \omega}} \int \Psi^*(x, t) (\hat{a}_+ + \hat{a}_-) \Psi(x, t) dx
\]
\[
= \frac{1}{2} \sqrt{\frac{\hbar}{2m \omega}} \int \left( \psi_p^* e^{iE_p t/h} + e^{-i\phi} \psi_{p+1}^* e^{iE_{p+1} t/h} \right) \left( \hat{a}_- + \hat{a}_+ \right) \psi_p e^{-iE_p t/h} + e^{i\phi} \psi_{p+1} e^{-iE_{p+1} t/h} dx
\]
\[
= \frac{1}{2} \sqrt{\frac{\hbar}{2m \omega}} \int \left( \psi_p^* \psi_p e^{-iE_p t/h} + e^{-i\phi} \psi_{p+1}^* \psi_{p+1} e^{-iE_{p+1} t/h} \right)
\]
\[
\cdot \left( \sqrt{p+1} \psi_{p+1} e^{-iE_{p+1} t/h} + \sqrt{p+2} \psi_{p+2} e^{-iE_{p+2} t/h} \right) dx
\]
\[
= \frac{1}{2} \sqrt{\frac{\hbar}{2m \omega}} \int \left( \psi_p^* \psi_p e^{-iE_p t/h} + e^{-i\phi} \psi_{p+1}^* \psi_{p+1} e^{-iE_{p+1} t/h} \sqrt{p+1} \psi_{p+1} e^{-iE_{p+1} t/h} + e^{i\phi} \sqrt{p+2} \psi_{p+2} e^{-iE_{p+2} t/h} \right) dx
\]
\[
= \frac{1}{2} \sqrt{\frac{\hbar}{2m \omega}} \sqrt{p+1} \left( e^{i\phi} e^{i(E_p-E_{p+1}) t/h} + e^{-i\phi} e^{-i(E_p-E_{p+1}) t/h} \right) = \sqrt{\frac{\hbar(p+1)}{2m \omega}} \cos(\omega t - \phi)
\]
where the last step used the fact that $E_p - E_{p+1} = \hbar \omega$.

(d) The classical equation of motion for a harmonic oscillator is $x = A \cos(\omega t + \beta)$, where $\beta$ is a constant and $A = \sqrt{2E_{max}/m\omega^2}$. Compare this to your answer to part (c). Can you think of a physical explanation for any differences? (Hint: think about the meaning of the expectation value $\langle x \rangle$.)

The quantum expectation value can be written

$$\langle \hat{x} \rangle = \sqrt{\frac{\hbar}{2m\omega^2}} \cos(\omega t - \phi)$$

The frequency is the same as the classical frequency, and the arbitrary classical phase translates into an arbitrary quantum phase. But the quantum amplitude is only half the classical one. $\langle \hat{x} \rangle$ is an average—even if the particle has some probability of reaching its classical maximum position, its position is “smeared out,” lowering the average. This can also be seen by noting that in the probability density $|\Psi|^2$, only the interference terms—the terms involving products of $\psi_p$ and $\psi_{p+1}$—give a time-dependent contribution; the motion is superposed on a time-independent piece $|\psi_p|^2 + |\psi_{p+1}|^2$.

2. **Square well evolution:**

Consider an infinite square well of width $a$. Suppose that at time $t = 0$, the wave function is $\Psi(x, 0) = A$

where $A$ is a constant.

(a) Find $A$.

$$1 = \int |\Psi(0)|^2 dx = \int_0^a |A|^2 dx = |A|^2 a$$

so $|A| = 1/\sqrt{a}$. We can take $A$ to be real, since an overall phase for the total wave function is unobservable. So $A = 1/\sqrt{a}$.

(b) Find $\Psi(x, t)$ at an arbitrary time $t$. See part I, question 5 for the procedure.

$$\Psi(x, 0) = \sum_n c_n \psi_n(x)$$

$$c_n = \int \Psi_n^* \Psi(x, 0) dx = \sqrt{\frac{2}{a}} \int_0^a \sin \left( \frac{\pi n x}{a} \right) \Psi(x, 0) dx = \sqrt{\frac{2}{a}} \int_0^a \sin \left( \frac{\pi n x}{a} \right) dx$$

$$= -\frac{\sqrt{2}}{\pi n} \cos \left( \frac{\pi n x}{a} \right) \bigg|_0^a = -\frac{\sqrt{2}}{\pi n} (\cos \pi n - 1) = \begin{cases} \frac{2\sqrt{2}}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

So

$$\Psi(x, t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar} = \sum_{n \text{ odd}} \frac{4}{\pi n \sqrt{a}} \sin \left( \frac{\pi n x}{a} \right) e^{-iE_n t/\hbar}$$

with $E_n = \frac{n^2 \hbar^2 \pi^2}{2ma^2}$. 
3. Commutators for the harmonic oscillator:
Recall that the commutator of two operators \( \hat{A} \) and \( \hat{B} \) is defined by the condition that
\[
[\hat{A}, \hat{B}] f = \hat{A} \hat{B} f - \hat{B} \hat{A} f
\]
for any function \( f \). A fundamental premise in quantum mechanics is that
\[
[\hat{x}, \hat{p}] = i\hbar
\]
(a) For a harmonic oscillator, find the commutator \([\hat{a}_+, \hat{a}_-]\). Start with the expressions for \( \hat{a}_\pm \) in terms of \( \hat{x} \) and \( \hat{p} \). Show your work!

Note first that \([A + B, C] = [A, C] + [B, C] \) and \([A, C + D] = [A, C] + [A, D] \).
\[
\hat{a}_\pm = \frac{1}{\sqrt{2m\hbar}} (\mp i\hat{p} + m\omega \hat{x})
\]
\[
[\hat{a}_+, \hat{a}_-] = \frac{1}{2m\hbar \omega} [-i\hat{p} + m\omega \hat{x}, i\hat{p} + m\omega \hat{x}]
\]
\[
= \frac{1}{2m\hbar \omega} ([-i\hat{p}, i\hat{p}] + [m\omega \hat{x}, m\omega \hat{x}] + [m\omega \hat{x}, i\hat{p}] + [m\omega \hat{x}, m\omega \hat{x}])
\]
\[
= \frac{1}{2m\hbar \omega} (0 - im\hbar[\hat{p}, \hat{x}] + im\hbar[\hat{x}, \hat{p}] + 0)
\]
\[
= \frac{1}{2m\hbar \omega} \cdot 2im\omega[\hat{x}, \hat{p}] = \frac{1}{2m\hbar \omega} \cdot 2im\omega \cdot i\hbar = -1
\]
(b) Find the commutator \([\hat{H}, \hat{a}_+]\). Again, show your work!

\[
\hat{H} = \hbar \omega \left( \hat{a}_+ \hat{a}_- + \frac{1}{2} \right)
\]
\[
[\hat{H}, \hat{a}_+] = [\hbar \omega \hat{a}_+ \hat{a}_-, \hat{a}_+] + \left[ \frac{\hbar \omega}{2}, \hat{a}_+ \right] = [\hbar \omega \hat{a}_+ \hat{a}_-, \hat{a}_+] = \hbar \omega [\hat{a}_+ \hat{a}_-, \hat{a}_+]
\]

But
\[
[\hat{a}_+ \hat{a}_-, \hat{a}_+] = \hat{a}_+ \hat{a}_- \hat{a}_+ - \hat{a}_+ \hat{a}_- \hat{a}_+ = \hat{a}_+ [\hat{a}_-, \hat{a}_+] = \hat{a}_+
\]
from part (a) (note that \([\hat{a}_-, \hat{a}_+] = -[\hat{a}_+, \hat{a}_-] \)). So
\[
[\hat{H}, \hat{a}_+] = \hbar \omega \hat{a}_+
\]
Alternatively, one could also write
\[
\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m\omega^2 \hat{x}^2
\]
\[
[\hat{H}, \hat{a}_+] = \frac{1}{2m} \left[ \hat{p}^2 + \frac{1}{\sqrt{2m\hbar \omega}} (-i\hat{p} + m\omega \hat{x}) \right] + \frac{1}{2} m\omega^2 \left[ \hat{x}^2, \frac{1}{\sqrt{2m\hbar \omega}} (-i\hat{p} + m\omega \hat{x}) \right]
\]
\[
= \frac{1}{2m} \cdot \frac{1}{\sqrt{2m\hbar \omega}} [\hat{p}^2, m\omega \hat{x}] + \frac{1}{2} m\omega^2 \cdot \frac{1}{\sqrt{2m\hbar \omega}} [\hat{x}^2, -i\hat{p}]
\]
But
\[
[\hat{p}^2, \hat{x}] = \hat{p}^2 \hat{x} - \hat{x} \hat{p}^2 = \hat{p}^2 \hat{x} - \hat{x} \hat{p} \hat{p} + \hat{p} \hat{x} \hat{p} - \hat{x} \hat{p}^2 = \hat{p} [\hat{x}, \hat{p}] + [\hat{x}, \hat{p}] \hat{p} = -2i\hbar \hat{p}
\]
\[
[\hat{x}^2, \hat{p}] = \hat{x}^2 \hat{p} - \hat{p} \hat{x}^2 = \hat{x}^2 \hat{p} - \hat{x} \hat{p} \hat{x} + \hat{x} \hat{p} \hat{x} - \hat{p} \hat{x}^2 = \hat{x} [\hat{x}, \hat{p}] + [\hat{x}, \hat{p}] \hat{x} = 2i\hbar \hat{x}
\]
\[
[H, \hat{a}_\pm] = \frac{1}{\sqrt{2m\hbar\omega}} \left\{ \frac{1}{2m} \cdot m\omega (-2i\hbar\hat{p}) + \frac{1}{2} m\omega^2 \cdot (-i) (2i\hbar\hat{x}) \right\} 
\]
\[
= \frac{1}{\sqrt{2m\hbar\omega}} \left\{ -i\hbar\omega\hat{p} + \hbar m\omega^2 \hat{x} \right\} = \hbar\omega\hat{a}_\pm
\]

(c) Find the expectation value \( \langle [\hat{a}_+, \hat{a}_-] \rangle_n \) in an energy eigenstate \( \psi_n \) by directly computing the action of the operators \( \hat{a}_\pm \) on the harmonic oscillator wave functions. Compare your result to that of part (a).

\[
\langle [\hat{a}_+, \hat{a}_-] \rangle_n = \int \psi_n^*(\hat{a}_+ \hat{a}_- - \hat{a}_- \hat{a}_+) \psi_n dx 
\]
\[
= \int \psi_n^* (\hat{a}_+ \sqrt{n} \psi_{n-1} - \hat{a}_- \sqrt{n+1} \psi_{n+1}) dx 
\]
\[
= \int \psi_n^* (\sqrt{n} \sqrt{n} \psi_n - \sqrt{n+1} \sqrt{n+1} \psi_n) dx = n - (n+1) = -1
\]

This agrees with (a), since \( \langle -1 \rangle = -1 \).

4. **Infinite potential barrier:**

A particle moves in a potential with an infinite barrier,

\[
V(x) = \begin{cases} 
0 & x < 0 \\
\infty & x > 0 
\end{cases}
\]

(a) Find the energy eigenstates.

The potential is zero for \( x < 0 \), so time-independent Schrödinger equation is

\[
-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E\psi \quad (x < 0)
\]

with boundary conditions—as in the infinite well—that \( \psi(0) = 0 \). Writing the Schrödinger equation as

\[
\frac{d^2 \psi}{dx^2} + k^2 \psi = 0 \quad \text{with} \quad k^2 = \frac{2mE}{\hbar^2} \quad (x < 0)
\]

we see that the general solution is of the form \( \psi(x) = A \cos kx + B \sin kx \). The boundary conditions are \( \psi(0) = 0 = A \). So the energy eigenstates are

\[
\psi(x) = B \sin kx, \quad E = \frac{\hbar^2 k^2}{2m}
\]

(b) Far to the right of a any potential barrier, the wave function takes the form

\[
\psi(x) = e^{ikx} + re^{-ikx}
\]

where \( r \) is a constant. The reflection coefficient \( R = |r|^2 \) then gives the probability that the particle was reflected from the barrier. What is the probability of reflection from the infinite potential barrier of part (a)?
$$\psi(x) = B \sin kx = \frac{B}{2i} \left( e^{ikx} - e^{-ikx} \right)$$

We can match the desired form by choosing $B = 2i$. Then

$$\psi(x) = e^{ikx} - e^{-ikx}$$

so $r = -1$. Therefore $R = r^2 = 1$, and the reflection probability is one (the particle is always reflected).