1. Simultaneous eigenvectors: an example (10 points)
Consider the two matrices
\[
A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}
\]

a. Find the eigenvector(s) and eigenvalues of \( A \).
b. Suppose that the eigenvectors you found in part a are also eigenvectors of \( B \). Find what restrictions this places on \( B \). Show that for \( A \) and \( B \) to have simultaneous eigenvectors, the commutator \( [A, B] \) must be zero.

2. Simultaneous eigenvectors in general (10 points)
Let \( \hat{A} \) and \( \hat{B} \) be two hermitian operators. A simultaneous eigenstate of \( \hat{A} \) and \( \hat{B} \) is a state obeying
\[
\hat{A}\ket{\psi_n} = a_n\ket{\psi_n}, \quad \hat{B}\ket{\psi_n} = b_n\ket{\psi_n}
\]
Show that if \( \hat{A} \) and \( \hat{B} \) do not commute, they cannot have a complete set of simultaneous eigenstates.
(Hint: look at \( [\hat{A}, \hat{B}]\ket{\psi_n} \) for any particular simultaneous eigenstate, and then think about what this would imply for a complete set.)

3. Expectation values (10 points)
Let \( \hat{A} \) be a hermitian operator, with a complete set of eigenstates \( \ket{a_n} \). The expectation value of \( \hat{A} \) in a state \( \ket{\psi} \) is defined to be
\[
\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle
\]
Show that
\[
\langle \hat{A} \rangle = \sum_n a_n |\langle a_n | \psi \rangle|^2
\]

4. Matrix elements for a harmonic oscillator (10 points)
Let \( \ket{n} \) denote the energy eigenfunctions of the harmonic oscillator.
a. Find the matrix elements \( \langle n|\hat{x}|n' \rangle \) and \( \langle n|\hat{p}|n' \rangle \), and write these as infinite matrices. (Show enough of each matrix to demonstrate the pattern.)
b. Using ordinary matrix multiplication, compute the commutator of the two matrices in part a.

5. Virial theorem (10 points)
Show that
\[
\frac{d}{dt}\langle \hat{x}\hat{p} \rangle = 2\langle \hat{T} \rangle - \left< \frac{dV}{dx} \right>
\]
where \( T \) and \( V \) are the kinetic and potential energies. In a stationary state, the left-hand side is zero: explain why. The resulting relation between \( T \) and \( V \) is called the virial theorem, and is important in many fields of physics—it tells us how energy is divided up among various components of a system.