Physics 115A      Spring 2005

Problem Set 6
Due by 4:00 pm Wednesday, 5/31 (in class, at my office, or in my mailbox)

1. Simultaneous eigenvectors: an example (10 points)
   Consider the two matrices
   \[
   M_1 = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad M_2 = \begin{pmatrix} d & e \\ f & g \end{pmatrix}
   \]
   with \(a \neq c\)
   
   a. Find the eigenvectors and eigenvalues of \(M_1\). (There are two different eigenvectors; be sure you find both.)
   
   b. Suppose that both of the eigenvectors you found in part a are also eigenvectors of \(M_2\).
   Find what restrictions this places on \(M_2\). Show that for \(M_1\) and \(M_2\) to have two simultaneous eigenvectors, the commutator \([M_1, M_2]\) must be zero.

2. Simultaneous eigenvectors in general (10 points)
   Let \(\hat{A}\) and \(\hat{B}\) be hermitian operators. A simultaneous eigenstate is a state obeying
   \[
   \hat{A}|\psi_n\rangle = a_n|\psi_n\rangle, \quad \hat{B}|\psi_n\rangle = b_n|\psi_n\rangle
   \]
   Show that if \(\hat{A}\) and \(\hat{B}\) do not commute, they cannot have a complete set of simultaneous eigenstates. (Hint: look at \([\hat{A}, \hat{B}]|\psi_n\rangle\) for any particular simultaneous eigenstate, and then think about what this would imply for a complete set.)

3. Expectation values (10 points)
   Let \(\hat{A}\) be a hermitian operator, with a complete set of eigenstates \(|a_n\rangle\). The expectation value of \(\hat{A}\) in a state \(|\psi\rangle\) is defined to be
   \[
   \langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle
   \]
   Show that
   \[
   \langle \hat{A} \rangle = \sum_n a_n |\langle a_n | \psi \rangle|^2
   \]

4. Virial theorem (10 points)
   Show that
   \[
   \frac{d}{dt} \langle \hat{x} \hat{p} \rangle = 2\langle \hat{T} \rangle - \langle x \frac{dV}{dx} \rangle
   \]
   where \(T\) and \(V\) are the kinetic and potential energies. In a stationary state, the left-hand side is zero: explain why. The resulting relation between \(T\) and \(V\) is called the virial theorem, and is important in many fields of physics—it tells us how energy is divided up among various components of a system.

5. Hermitian operators (10 points)
   a. The operators \(\hat{A}, \hat{B},\) and \(\hat{C}\) are all hermitian, with \([\hat{A}, \hat{B}] = \hat{C}\). Show that \(\hat{C} = 0\).
   
   b. The operator \(\hat{F}\) is defined by
   \[
   \hat{F}\psi(x) = \psi(x + a) + \psi(x - a)
   \]
   Determine whether \(\hat{F}\) is hermitian.