1. Finite barrier (20 points)
Consider the finite potential barrier

\[ V(x) = \begin{cases} 
V_0 & \text{for } -a < x < a \\
0 & \text{for } |x| > a 
\end{cases} \]

Suppose a particle approaches the barrier from the left with energy \( E \).

a. Compute the transmission and reflection probabilities for \( E < V_0 \).

This is easiest if you shift the origin, so the barrier goes from 0 to 2\( a \), and write the wave function in region II in terms of hyperbolic functions (sinh and cosh) rather than exponentials. Let

\[ \frac{2mE}{\hbar^2} = k^2, \quad \frac{2m(E - V_0)}{\hbar^2} = -\kappa^2 \]

Then the wave function is

\[ \psi(x) = \begin{cases} 
e^{ikx} + re^{-ikx} & \text{for } x < 0 \\
C \sinh \kappa x + D \cosh \kappa x & \text{for } 0 < x < 2a \\
te^{ikx} & \text{for } x > 2a 
\end{cases} \]

Define the abbreviations

\[ \beta = \frac{k}{\kappa} = \sqrt{\frac{E}{V_0 - E}}, \quad c = \cosh 2\kappa a, \quad s = \sinh 2\kappa a \]

Now impose continuity of \( \psi \) and \( d\psi/dx \) at \( x = 0 \) and \( x = 2a \):

\[ \psi(0) = 1 + r = D \quad (1) \]
\[ \psi'(0) = ik(1 - r) = \kappa C \quad (2) \]
\[ \psi(2a) = Cs + Dc = te^{2ika} \quad (3) \]
\[ \psi'(2a) = \kappa(Cc + Ds) = ikte^{2ika} \quad (4) \]

Eqns. (1) and (2) give \( D = 1 + r \) and \( C = i\beta(1 - r) \). Combining (3) and (4),

\[ i\beta te^{2ika} = Cc + Ds = i\beta(Cs + Dc) \]

Inserting the values of \( C \) and \( D \) into this equality, and doing a little algebra,

\[ r = \frac{(1 + \beta^2)s}{2i\beta c - (1 - \beta^2)s} \quad (5) \]

Inserting this into (4), a little more algebra gives

\[ t = \frac{2i\beta e^{-2ika}}{2i\beta c - (1 - \beta^2)s} \quad (6) \]
Then taking the absolute square, and using the fact that $\beta$, $c$, and $s$ are real, we get a transmission probability

$$T = |t|^2 = \frac{4\beta^2}{4\beta^2c^2 + (1 - \beta^2)s^2} = \frac{4\beta^2}{4\beta^2 + (1 + \beta^2)s^2}$$

where the last line used $c^2 - s^2 = \cosh^2 2\kappa a - \sinh^2 2\kappa a = 1$. The reflection probability is

$$R = |r|^2 = \frac{(1 + \beta^2)s^2}{4\beta^2 + (1 + \beta^2)s^2} = 1 - T$$

b. Compute the transmission and reflection probabilities for $E > V_0$.

Since $E - v_0$ is now positive, define

$$\frac{2m(E - V_0)}{\hbar^2} = \ell^2$$

We could redo the whole calculation, but instead notice that up to the computation of $r$ and $t$, the math is identical to part a, but with $\kappa = il$. Make the redefinitions

$$\beta = \frac{k}{il} = -i\hat{\beta}, \quad c = \cosh 2ila = \cos 2la = \hat{c}, \quad s = \sinh 2ila = i \sin 2la = i\hat{s}$$

where $\hat{\beta} = k/l$, $\hat{c} = \cos 2la$, and $\hat{s} = \sin 2la$. Substituting into eqns. (5) and (6),

$$\hat{r} = \frac{i(1 - \hat{\beta}^2)\hat{s}}{2\beta\hat{c} - i(1 + \hat{\beta}^2)\hat{s}}$$

$$\hat{t} = \frac{2\beta e^{-2\kappa a}}{2\beta\hat{c} - i(1 + \hat{\beta}^2)\hat{s}}$$

Taking the absolute squares,

$$\hat{T} = |\hat{t}|^2 = \frac{4\hat{\beta}^2}{4\hat{\beta}^2\hat{c}^2 + (1 + \hat{\beta}^2)\hat{s}^2} = \frac{4\hat{\beta}^2}{4\hat{\beta}^2 + (1 - \hat{\beta}^2)\hat{s}^2}$$

where in the last line I’ve used $\hat{s}^2 + \hat{c}^2 = 1$, and

$$\hat{R} = |\hat{r}|^2 = \frac{(1 - \hat{\beta}^2)\hat{s}^2}{4\hat{\beta}^2 + (1 - \hat{\beta}^2)\hat{s}^2} = 1 - \hat{T}$$

c. For what values of $E$ is the barrier “totally transparent”?

For total transparency, we need $R = 0$ (no reflection). For part a, this is never true, since $s = \sinh 2\kappa a = 0$ only for $\kappa = 0$, i.e., $E = V_0$. For part b, though, $R = 0$ if $\hat{s} = \sin 2la = 0$, that is,

$$2la = 2a \sqrt{\frac{2m(E - V_0)}{\hbar}} = n\pi, \quad E = V_0 + \frac{n^2\pi^2\hbar^0}{8ma^2}$$
2. **Infinite square well with a delta function** (15 points)

Consider an infinite square well with a delta function potential in the middle, that is,

\[ V(x) = \begin{cases} 
\delta \left( x - \frac{a}{2} \right) & \text{for } 0 < x < a \\
\infty & \text{elsewhere}
\end{cases} \]

Find the bound state energies. (You will get a transcendental equation of the same type Griffiths describes for the finite square well. Don’t try to solve it exactly, but describe the solutions qualitatively. How do the energies compare with those for the ordinary infinite well?)

Let \( \frac{2mE}{\hbar^2} = k^2 \). The wave function to the left and right of \( x = a/2 \) will be a combination of \( \sin kx \) and \( \cos kx \), and must be zero at \( x = 0 \) and \( x = a \). So we can write

\[ \psi(x) = \begin{cases} 
A \sin kx & \text{if } 0 < x < \frac{a}{2} \\
B \sin (k(x - a)) & \text{if } \frac{a}{2} < x < a \\
0 & \text{elsewhere}
\end{cases} \]

[Note that \( \sin k(x - a) = \sin kx \cos ka - \cos kx \sin ka \), so the wave function in the region \( a/2 < x < a \) has the right form as a linear combination; this form was chosen so that \( \psi(a) = 0 \).]

Continuity at \( x = a/2 \):

\[ \psi(a/2) = A \sin ka/2 = B \sin(-ka/2) \]

so either \( A = -B \) or \( \sin ka/2 = 0 \). [The second possibility is tricky – I missed it first time through!]

Derivative condition:

\[
\int_{a/2-\epsilon}^{a/2+\epsilon} \left[ \frac{d^2 \psi}{dx^2} + \frac{2m}{\hbar^2} \left( E - \delta \left( x - \frac{a}{2} \right) \right) \right] \psi \, dx = 0 = \left. \frac{d\psi}{dx} \right|_{a/2-\epsilon}^{a/2+\epsilon} - \frac{2m}{\hbar^2} \psi(a/2)
\]

so

\[ Bk \cos(-ka/2) - Ak \cos ka/2 = \frac{2m}{\hbar^2} A \sin ka/2 \]

• If \( B = -A \), we obtain \( k \cos ka/2 = (m/\hbar^2) \sin ka/2 \).
• If \( \sin ka/2 = 0 \), we obtain \( (B - A)k \cos ka/2 = 0 \), or \( B = A \).

Thus

\[ \text{either } \quad k \cot \frac{ka}{2} = -\frac{m}{\hbar^2} \quad \text{or} \quad \sin \frac{ka}{2} = 0 \]

The second condition gives \( ka = 2\pi n \), which yields energy levels exactly equal to those the ordinary infinite square well levels with even quantum numbers,

\[ E_n = \frac{\pi^2 \hbar^2 (2n)^2}{2ma^2} \]
The first condition can’t be solved exactly, but for $k$ large, the solutions are

$$ka \approx (2n + 1)\pi + \frac{2m}{\pi}h^2(2n + 1), \quad E \approx \frac{\pi^2}{2ma^2} + \frac{2}{a}$$

which are near to the remaining infinite square well states. (This approximate solution comes from noting that if $k$ is large, the condition requires that $\cot ka/2$ be small. $\cot x = 0$ for $x = (n + \frac{1}{2})\pi$, so you can write $\frac{ka}{2} = (n + \frac{1}{2})\pi + \epsilon$ and expand in $\epsilon$.)

3. Leaky well (15 points)
Consider the potential

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ \alpha\delta(x - a) & \text{for } x \geq 0 \end{cases}$$

(Sketch this potential; otherwise you are likely to get confused!) Consider a particle that starts in the “well” $0 < x < a$ and slowly “leaks out” through the barrier. Note that this means that to the right of the delta function, the particle will be moving only in the positive $x$ direction.

a. Solve the time-independent Schrödinger equation for these boundary conditions, and find an implicit equation for the “energy” $E$.

As usual, let $\frac{2mE}{\hbar^2} = k^2$. The boundary conditions require that $\psi(0) = 0$ and that $\psi \sim e^{ikx}$ for $x > a$, so

$$\psi(x) = \begin{cases} 0 & \text{if } x < 0 \\ A\sin kx & \text{if } 0 < x < a \\ Be^{ikx} & \text{if } x > a \end{cases}$$

Continuity at $a$:

$$\psi(a) = A\sin ka = Be^{ika} \Rightarrow B = Ae^{-ika}\sin ka$$

Derivative condition (see preceding problem):

$$\frac{d\psi}{dx}\bigg|_{a+} - \frac{2ma}{\hbar^2} \psi(a) = 0 = ikBe^{ika} - kA\cos ka - \frac{2ma}{\hbar^2} A\sin ka$$

Substituting the value of $B$, we get

$$\left(ik - \frac{2ma}{\hbar^2}\right) = k\cot ka$$

b. Show that the equation you end up with for $E$ will have solutions only for complex “energies.”

If $k$ is real, $k\cot ka$ is real, but the left-hand side of the equation above is complex, so there is no solution. Thus $k$ must be complex, and so must $E = \hbar^2k^2/2m$.

c. Explain the significance of a complex $E$. In particular, if you write $E = E_0 + i\Gamma$, what is the time dependence of the probability density $|\psi|^2$ for an energy eigenfunction? What does that mean for this problem?
For an energy eigenstate, such as the ones we are considering,

$$\Psi(x, t) = \psi(x)e^{-iEt/h} = \psi(x)e^{-iE_0t/\hbar}e^{\Gamma t/\hbar}$$

The probability density is therefore

$$|\Psi(x, t)|^2 = |\psi(x)|^2 e^{2\Gamma t/\hbar} = |\Psi(x, 0)|^2 e^{2\Gamma t/\hbar}$$

Γ had better be negative, since the probability has to stay less than or equal to one. For Γ < 0,

$$|\Psi(x, t)|^2 |\Psi(x, 0)|^2 e^{-2|\Gamma|t/\hbar}$$

which describes exponential decay (hence the “leaky” well in this problem). The half-life—the time it takes for the probability to fall by a factor of 2—is determined by

$$e^{-2|\Gamma|t/\hbar} = \frac{1}{2} \Rightarrow t_{1/2} = \frac{\hbar \ln 2}{2|\Gamma|}$$